Massive vector fields in Loop Quantum Gravity

Ph. D. Thesis

Gábor Helesfai

Department of Theoretical Physics

Eötvös Loránd University, Budapest

Supervisor: Dr. Gyula Bene

Graduate School for Physics

Doctorial Program for Particle Physics and Astronomy

Head of School: Prof. Dr. Zalán Horváth

Head of Program: Prof. Dr. Ferenc Csikor

Contents

Acknowledgements

I would like to thank my supervisor Dr. Gyula Bene his advises and help in my work. I would also like to thank Prof. Jerzy Lewandowsky for giving me opportunity to discuss important issues about by research and have visits at Warsaw University. Last but not least I wish to thank Benjamin Beri, Janos Majar and Adam Rusznyak for their hints and tips regarding my work.

Notation

M - 4 dimensional Lorentz manifold

σ - 3 dimensional Riemann manifold

$\mu, \nu, \ldots$ - spacetime indices, taking values from 0,1,2,3

$a, b, \ldots$ - spatial indices, taking values from 1,2,3

$I, J, \ldots$ - internal/SU(2) indices, taking values from 1,2,3

∇_μ - covariant derivative on the manifold M

D_a - covariant derivative on the manifold Σ

G - gravitational constant

$\tau_I = i\sigma_I$ - generators of $SU(2)$ (σ_I are the Pauli-matrices)

$\hbar$ - Planck constant

Q - Yang-Mills coupling constant

$l_P = \sqrt{\hbar\kappa}$ - Planck-length

$m_P = \sqrt{\hbar/\kappa}$ - Planck-mass

$\alpha_Q = \hbar Q^2$ - fine structure constant

N - lapse function

N_a - shift vector

H - Hamiltonian/scalar constraint

H_a - diffeomorphism constraint

G - SU(2) Gauss/gauge constraint

$\underline{G}$ - U(1) Gauss/gauge constraint

E^a_i - electric field for the gravitational field

A^a_i - Ashtekar-connection for the gravitational field

$\underline{E}^a$ - electric field for the U(1) Yang-Mills field

$\underline{A}^a$ - connection for the U(1) Yang-Mills field

Φ - scalar field

Π - canonical momentum for the scalar field

γ - graph

$e_1, \ldots, e_j$ - edges of the graph

$v_1, \ldots, e_k$ - vertices of the graph

$h_e(A)$ *or* $A(e)$ - holonomy for the Ashtekar variable along the edge e

$E(S)$ - electric flux of the electric field for a surface S

$\underline{h}_e(\underline{A})$ - holonomy for the electromagnetic field along the edge e

$\underline{E}(S)$ - electric flux of the electromagnetic field for a surface S

$U(\Phi(v), \lambda)$ - point-holonomy for the scalar field at vertex v

$\Pi(B)$ - smeared momentum of the scalar field for an open ball $B \subset \sigma$.

$T_{\gamma,\vec{\pi},\vec{I}}(A)$ - spin network function

$F_{\gamma,\vec{n}}(\underline{A})$ - flux network state

$D_{\gamma,\vec{\lambda}}(\Phi)$ - vertex function

$\mathcal{A}$ - space of generalized connections for the Ashtekar connection

$\underline{\mathcal{A}}$ - space of generalized connections for the electromagnetic connection

$\mathcal{U}$ - space of generalized Higgs fields (real valued functions)

μ - measure

$\mathcal{H}^{GR}, \mathcal{H}^{EM}, \mathcal{H}^{SK}$ - Hilbert space for the gravitational, electromagnetic and scalar fields respectively

1 Introduction

Since the fundamental principles of quantum field theory were laid down, many efforts were made to apply these to general relativity. Unfortunately these were not successful for several reasons. In the first place, the methods of standard perturbative quantum field theory cannot be applied because the coupling constant is not dimensionless, resulting in a non-renormalizable theory. Hence, alternative approaches are needed. A major difficulty arises because the Einstein equations are nonlinear differential equations for the metric. This is a serious problem since even if one defines a suitable Hilbert-space, the definition of an operator representing e.g. $\frac{1}{\sqrt{det(-g)}}$ seems extremely difficult, and until recently one could not solve it through the Hamiltonian framework. This difficulty have been solved by string theory in a perturbative framework, however, string theory does not explicitly satisfy the requirement of diffeomorphism covariance. This is an important question, since diffeomorphism covariance is the main element of general relativity and as such, one would like to implement this at the quantum level. The goal of loop quantum gravity is to solve both problems. This seems to be quite a difficult task, since in order not to break diffeomorphism covariance, one cannot use perturbative methods. If one wanted to summarize, *Loop Quantum Gravity is an attempt to construct a non-perturbative, background independent*[1] *quantization of general relativity.* The present thesis focuses entirely on applying Loop Quantum Gravity. For completeness, however we mention alternative approaches.

1. The twistor theory [5], originally developed by Roger Penrose in 1967, is the mathematical theory which maps the geometric objects of the four dimensional space-time (Minkowski space) into the geometric objects in the 4-dimensional complex space with the metric signature (2,2). The coordinates in such a space are called "twistors." The motivation was the generalization of spin networks: it was believed that new quantum object must combine angular momentum (spin) with linear momentum, and on an equal footing.

2. Supergravity [6] is a field theory that combines the principles of supersymmetry and general relativity. Together, these imply that, in supergravity, the supersymmetry is a local symmetry (in contrast to non-gravitational supersymmetric theories, such as the Minimal Supersymmetric Standard Model). The theory of supergravity contains a

[1] The concept of background independence will be explained in detail in section 3.1

spin-2 field whose quantum is the graviton. Supersymmetry requires the graviton field to have a superpartner. This field has spin 3/2 and its quantum is the gravitino. The number of gravitino fields is equal to the number of supersymmetries.

3. The program of causal sets [7] is an approach to quantum gravity where spacetime is fundamentally discrete and that the spacetime events are related by a partial order. This partial order has the physical meaning of the causality relations between spacetime events. The program is based on a theorem by David Malament which states that if there is a bijective map between two past and future distinguishing spacetimes which preserves their causal structure then the map is a conformal isomorphism. The conformal factor that is left undetermined is related to the volume of regions in the spacetime. This volume factor can be recovered by specifying a volume element for each spacetime point. The volume of a spacetime region could then be found by counting the number of points in that region. Causal sets was initiated by Rafael Sorkin who continues to be the main proponent of the program. The program provides a theory in which spacetime is fundamentally discrete while retaining local Lorentz invariance.

4. Non-commutative geometry [8] is a branch of mathematics concerned with the possible spatial interpretations of algebraic structures for which the commutative law fails. The challenge of NCG theory is to get around the lack of commutative multiplication, which is a requirement of previous geometric theories of algebraic structures.

5. String theory [4] has a good chance to be a theory that unifies all interactions. Originally it was a two-dimensional field theory of world-sheets embedded into fixed, D-dimensional pseudo-Riemann manifold. If one perturbates the metric $g = \eta + h$ and keeps only the lowest order one obtains a free field theory in two dimensions which is only consistent in D=26 (bosonic string) or D=10 (superstring). One of the most important results of the theory is that if one keeps higher order terms the construction is consistent if and only the metric satisfies the Einstein equations. Though the theory still has questions to be answered (extra dimensions, covariance), string theorists argue they have found a consistent theory for quantum general relativity.

6. Spin foam models are a path integral description of quantum general relativity. The basic structure here is the spin foam, which can be thought of as a discretizied version

of spacetime in a sense like the spin networks are discretizations of three dimensional space. Actually three dimensional slices of spin foams are spin networks. Unfortunately this theory is quite recent and as such there is no result that can either verify it or not.

The history of Loop Quantum Gravity began when Abhay Ashtekar reformulated Einstein's field equations of general relativity in 1986, using what have come to be known as Ashtekar variables [11]. In 1988, Carlo Rovelli and Lee Smolin used this formalism to introduce the loop representation [21] of quantum general relativity, which was soon developed by Ashtekar, Rovelli, Smolin and many others. In the Ashtekar formulation, the fundamental objects are the connection and the coordinate frame at each point. Because the Ashtekar formulation was background-independent, it was possible to use Wilson loops as the basis for a non-perturbative quantization of gravity. Explicit (spatial) diffeomorphism invariance plays an essential role in the regularisation of the Wilson loop states. Around 1990, Carlo Rovelli and Lee Smolin obtained an explicit basis of states of quantum geometry [24], which turned out to be labeled by Penrose's spin networks, and showed that the geometry is quantized [20], that is, the (non-gauge-invariant) quantum operators representing area and volume have a discrete spectrum. Shortly after Baez gave a precise definition for the spin network formalism [25]. Later Thiemann gave an alternative quantization for the volume operator [50] which was important for the Hamiltonian constraint to be well defined [33]. Matter coupling was beginning to be explored in 1994 by Morales and Rovelli [44]. They investigated the case of Fermions, which results were later extended by Thiemann [34] for electromagnetic and scalar fields. Current researches in this field include especially the question of anomalies of the constraint algebra [39], semi-classical analysis [40] (weaves) and loop quantum cosmology [13].

In the first few sections we will review the basic tools needed for this construction:

1. 3+1 decomposition: this is the first step, where one constructs the Hamiltonian formalism for general relativity (ADM formalism). It turns out that the Hamiltonian is linear combination of constraints.

2. Ashtekar variables: the new variables simplify the constraints of the ADM formalism and transforms the theory into a gauge theory.

3. Refined Algebraic Quantization (RAQ): this is the method which is used in Loop Quantum Gravity. It gives a precise prescription on how to quantize a gauge theory with

constraints, thus we can construct the operators and Hilbert space for the theory.

4. regularisation: in order to make quantization mathematically well-defined we need to smear out the Dirac-delta distributions appearing in the Poisson-algebra.

(For a detailed review see e.g. [36])
As mentioned above Loop Quantum Gravity is a non-perturbative quantization method of the gravitational field, which cannot be quantized perturbatively. Another field which cannot be quantized perturbatively is the Proca-field, i.e., the massive vector field. It should be added that a successful (renormalizable) perturbative quantization is possible through the Higgs mechanism. Note that the experimental verification of the existence of the Higgs-field is still to be awaited for. The question naturally arises whether the approach of Loop Quantum Gravity renders the quantization of the Proca-field possible. Moreover, one may quantize the vector field with a mass generated by the Higgs mechanism, by using the same approach. This makes possible the comparison of the two theories. Since our goal is to study the Proca field and spontaneous symmetry breaking, we will perform the steps of quantization with the presence of matter fields (vector and scalar fields). Though it is straightforward to extend the quantization procedure to these fields, there are subtle issues which need detailed attention (for example definition of the multiplication operator for the scalar field [45]).
After all the tools are available we are ready to study the massive fields, first the Proca field and after it spontaneous symmetry breaking. Even in the first case we will observe how different is this approach to the one used in conventional field theory. For instance we will find that gauge fixing is not only unnecessary in this case but it actually makes the whole construction more difficult. Instead we will actually solve the gauge constraint at the quantum level.
After that we turn to spontaneous symmetry breaking. Even at the classical level the similarities between this theory and the Proca field will be quite transparent, the only difference will be the origin of the mass. The new feature of spontaneous symmetry breaking will be that the mass will be a functional of a scalar field, thus in the quantum theory it will become an operator. By studying this operator we will be able to describe how one can interpret mass in this theory.
The thesis is organized as follows. In the first three chapters we will describe the basic tools of Loop Quantum Gravity: in chapter two we derive the Hamiltonian formalism and new

variables, in chapter three the basic elements of quantization and in chapter four the construction of the constraint operators. In chapter five and six we will turn to the Proca-field and spontaneous symmetry breaking, respectively.

2 Classical theory

We start with the classical theory of the Hamiltonian framework. First we introduce the ADM formalism, and after that the Ashtekar variables. At the end of the chapter the theory of a simple electromagnetic field coupled to a scalar field and gravity is investigated as well.

2.1 Hamiltonian formalism of general relativity

Since we want to quantize the theory in the Hamiltonian framework, first we need to derive the Legendre transformation for general relativity - this is the ADM (Arnowitt-Deser-Misner) formalism [9]. For general relativity it is a bit trickier than for other theories because the notion of time is not a trivial issue. This problem can be solved unambiguously for globally hyperbolic spacetimes, where the four dimensional spacetime manifold can be written in the following form: $M = \mathbb{R} \times \sigma$. In this case M is foliated with σ_t hypersurfaces , where t is a global "time function". We can introduce a nowhere vanishing, global "timeflow" vector field t^μ with the following property: $t^\mu \nabla_\mu t = 1$, where ∇_μ donates the covariant derivative on M. Now let us write this field in the in the following form:

$$t^\mu = Nn^\mu + N^\mu,$$

where N is called the lapse function, n^μ is the unit normal vector of σ, that is $n_\mu \sim \nabla_\mu t$ and $g_{\mu\nu}n^\mu n^\nu = -1$, and N^μ, called the shift vector, which is tangential to the hypersurface. Let us introduce the induced metric and the extrinsic curvature as

$$\begin{aligned} q_{\mu\nu} &:= g_{\mu\nu} + n_\mu n_\nu \\ K_{\mu\nu} &:= \frac{1}{2}\mathcal{L}_n q_{\mu\nu} \end{aligned}$$

where $g_{\mu\nu}$ is the four dimensional spacetime metric and $\mathcal{L}_n$ denotes Lie-derivative along n. It is easy to see that these quantities are spatial - i.e. they vanish when either of their indices are contracted with n_μ - so from now we can use latin indices in the case of q and K. Before we continue, it is worth to mention that the connections of the lapse function and shift vector with the metric are

$$N = -g_{\mu\nu}t^\mu n^\nu \tag{1}$$

$$N^{\mu} = q^{\mu}_{\nu} t^{\nu}. \tag{2}$$

To perform the 3+1 decomposition one needs to define the covariant derivative on Σ as

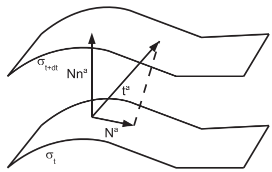

Figure 1: Illustration of the lapse function and shift vector.

$$D_{\mu} := q^{\nu}_{\mu} \nabla_{\nu}. \tag{3}$$

It can be seen that this derivative is the unique induced covariant derivative. Now we can substitute these quantities into the action

$$S = \frac{1}{8\pi G} \int d^4 x R \sqrt{-det(g)}, \tag{4}$$

where we can write R in the following form:

$$R = R^{(3)} + (K_{ab} K^{ab} - K^2) - 2\nabla_{\mu}(n^{\nu} \nabla_{\nu} n^{\mu} - n^{\mu} \nabla_{\nu} n^{\nu}). \tag{5}$$

which is called Codacci equation [3]. Here $K = tr(K_{ab})$ and $R^{(3)}$ is the induced curvature on σ, and the last term is a total differential and hence when one substitutes (5) into the

Lagrangian, it will lead to the following expression:

$$S = \int_{\mathbb{R}} dt \int_{\sigma} d^3x\mathcal{L} = \int_{\mathbb{R}} dt \int_{\sigma} d^3x\sqrt{det(q)}N[R^{(3)} + K^{ab}K_{ab} - K^2] \tag{6}$$

The extrinsic curvature is related to the "time derivative" $\dot{q}_{ab} = \mathcal{L}_t q_{ab}$ by

$$K_{ab} = \frac{1}{2}q_{ab} = \frac{1}{2N}\mathcal{L}_{t-N}q_{ab} = \frac{1}{2N}(\dot{q}_{ab} - D_aN_b - D_bN_a). \tag{7}$$

In the ADM-formalism the next step is to derive the canonical momenta for the variables q_{ab}, N_a, N:

$$P^{ab} = \frac{\delta S}{\delta \dot{q}_{ab}} = \frac{1}{8\pi G}\sqrt{det(q)}[K^{ab} - q^{ab}K] \tag{8}$$

$$P^a = \frac{\delta S}{\delta \dot{N}_a} = 0 \tag{9}$$

$$P = \frac{\delta S}{\delta \dot{N}} = 0, \tag{10}$$

respectively, which means that N and N_a are not dynamical variables, thus we have a singular Lagrangian (we cannot express $\dot{N}$ and $\dot{N}_a$ in terms of q_{ab}, N_a and N). The method dealing with singular Lagrangians has been developed by Dirac [10], and states that we must treat P^a and P as primary constraints, while we introduce appropriate Lagrange-multipliers λ_a and λ to them. After this we perform the usual Legendre-transformation on the remaining velocities. Before doing that let us check the consequence of the primary constraints. Since the Lagrangian is independent of $\dot{N}_a$ and $\dot{N}$ we can write

$$\frac{\partial \mathcal{L}}{\partial \dot{N}_a} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0 \tag{11}$$

Using the Euler-Lagrange equation we obtain

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{N}_a} = 0 = \frac{\partial \mathcal{L}}{\partial N_a} - \frac{\partial}{\partial x_b}\frac{\partial \mathcal{L}}{\partial \frac{\partial N_a}{\partial x_b}} \quad := H_a \tag{12}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{N}} = 0 = \frac{\partial \mathcal{L}}{\partial N} - \frac{\partial}{\partial x_b}\frac{\partial \mathcal{L}}{\partial \frac{\partial N}{\partial x_b}} \quad := H. \tag{13}$$

Since H and H_a are zero and since we had to use the equation of motion to derive them they are secondary constraints. They are called Hamiltonian (or scalar) and diffeomorphism constraints, respectively. To calculate them explicitly first we perform the derivations and then express $\dot{q}_{ab}$ with P_{ab}. To keep the calculations as simple as possible we use the following formulae:

$$\begin{aligned}
\dot{q}_{ab} &= 2NK_{ab} + (\mathcal{L}_{\vec{N}}q)_{ab} \\
\dot{q}_{ab}P^{ab} &= (\mathcal{L}_{\vec{N}}q)_{ab}P^{ab} + \frac{2N}{8\pi G}\sqrt{det(q)}(K_{ab}K^{ab} - K^2) \\
P_{ab}P^{ab} &= \frac{det(q)}{(8\pi G)^2}(K_{ab}K^{ab} + K^2) \\
(P^a_a)^2 &= \frac{4}{(8\pi G)^2}det(q)K^2.
\end{aligned}$$

This way we get the following form of the secondary constraints.

$$H = \frac{1}{\sqrt{det(q)}}(q_{ab}q_{cd} - \frac{1}{2}q_{ac}q_{bd})P^{ac}P^{bd} - \sqrt{det(q)}R \quad (14)$$

$$H_a = -2q_{ab}D_cP^{bc}. \quad (15)$$

Now we can perform the Legendre-transformation. Taking into account the previous results the Lagrangian density has the form

$$\mathcal{L} = [\dot{q}_{ab}P^{ab} - (N_aH^a + NH + \lambda_aP^a + \lambda P)], \quad (16)$$

thus obtaining the Hamiltonian

$$\bar{H} = \int(N_aH^a + NH + \lambda_aP^a + \lambda P) := H(N) + \vec{H}(\vec{N}) + P(\lambda) + \vec{P}(\vec{\lambda}) \quad (17)$$

The (non-trivial) Poisson brackets are:

$$\begin{aligned}
\{P^{ab}(t,x), q_{cd}(t,x')\} &= 8\pi G\delta^a_{(c}\delta^b_{d)}\delta(x - x') \\
\{P^a(t,x), N_b(t,x')\} &= 8\pi G\delta^a_b\delta(x - x') \\
\{P(t,x), N(t,x')\} &= 8\pi G\delta(x - x')
\end{aligned}$$

One can say that the Hamiltonian in general relativity is not a "true" Hamiltonian but a linear combination of primary and secondary constraints. One might worry that the equation of motion for H generate more constraints but the

$$\begin{aligned}
\{\vec{H}(\vec{f}), \vec{H}(\vec{f'})\} &= -8\pi G \vec{H}(\mathcal{L}_{\vec{f}}(\vec{f'})) \\
\{\vec{H}(\vec{f}), H(f')\} &= -8\pi G H(\mathcal{L}_{\vec{f}}(f')) \\
\{H(f), H(f')\} &= -8\pi G \vec{H}(q^{ab}(f f'_{,b} - f' f_{,b}))
\end{aligned}$$

Dirac algebra of the constraints prevents that (it is closed). This means that the constraint surface - i.e. where the constraints hold - is preserved under the motions generated by the constraints. This type of constraints is called first class constraints.

The first attempts to quantize general relativity used directly the ADM formalism as a starting point, but these were not successful for several reasons. The main problem was that though one can define corresponding operators to q_{ab} and P^{ab} it is hard to implement the Hamiltonian constraint because it is not only non-linear in the variable q_{ab} but not even analytic. In quantum field theory these operators become operator valued distributions which are divergent in cases where non-linear functions of these operators appear. Fortunately with the help of the Ashtekar-variables we are able to solve this problem.

2.2 The Ashtekar-variables

The motivation for introducing new variables was initially to simplify the constraints [11]. The concept was to cast the constraints into at least quadratic expressions, which provided a major step toward quantization. Though the first part of this program succeeded, it turned out that the quantization method developed cannot be applied in this case. However the "byproduct" of the method was that general relativity got the form of a gauge theory, thus one could apply the results attained in the field of classical and quantum gauge theory.

To introduce the Ashtekar variables, first let us define the triad fields e^I_a through the following relation:

$$q_{ab} := \delta_{JK} e^J_a e^K_b . \tag{18}$$

It is clear that the triads are not uniquely defined since the previous equation is invariant under local SO(3) (or SU(2)) rotations $e^I_a \to e^J_a O_{IJ}$. This increases the degrees of freedom which means that if we want this new results to be equivalent to the ADM formalism, we

need extra constraints (and as we will see later it is the case).
The spin connection (Γ^J_a) is defined via

$$D_a e^J_b := \partial_a e^J_b - \Gamma^c_{ab} e^J_c + \epsilon_{JKL} \Gamma^J_a e^L_b = 0, \tag{19}$$

where Γ^c_{ab} denotes the three dimensional Christoffel-symbol, ϵ_{JKL} is the Levi-Chivita symbol and D_a is the covariant derivative compatible to e^J_a. Just like the Christoffel-symbol with the metric, spin connection can be expressed in terms of e^I_a:

$$\Gamma^J_a = \frac{1}{2} \epsilon^{JKL} e^b_L (e^K_{a,b} - e^K_{b,a} + e^K_c e^M_a e^M_{c,b}). \tag{20}$$

Let us define K^I_a through the following expression:

$$2K_{ab} := sgn(det(e)) K^j_a e^j_b. \tag{21}$$

Since K_{ab} is symmetric, K^I_a must satisfy the condition

$$G_{ab} := K^j_{[a} e^j_{b]} = 0. \tag{22}$$

Instead of the triads we will use their dual tensor called the *electric field*:

$$E^a_J = \frac{1}{2} \epsilon^{abc} \epsilon_{JKL} e^K_b e^L_c. \tag{23}$$

Equations (22) and (20) can be expressed with the use of K^I_a and electric field yielding

$$\begin{aligned} G_{JK} &= K_{a[J} E^a_{K]} = 0 && (24) \\ \Gamma^J_a &= \frac{1}{2} \epsilon^{JKL} E^b_L (E^K_{a,b} - E^K_{b,a} + E^K_c E^M_a E^M_{c,b}) + \\ &+ \frac{1}{4} \epsilon^{JKL} E^b_L (2E^K_a \frac{det(E)_{,b}}{det(E)} - E^K_b \frac{det(E)_{,a}}{det(E)}) && (25) \end{aligned}$$

Now we can express the ADM variables q_{ab} and P^{ab} with E^I_a and K^I_a:

$$q_{ab} = E^J_a E^J_b |det(E)| \tag{26}$$

$$P^{ab} = \frac{1}{|det(E)|} E^a_J E^c_J K^L_{[c} \delta^b_{f]} E^f_L. \tag{27}$$

It is easy to see that if the constraint (24) is satisfied then the functions (27) reduce to the ADM coordinates. Inserting (27) into (14) and (15) we can write the Hamiltonian- and diffeomorphism constraints in terms of our new variables:

$$
\begin{aligned}
H_a &= -D_b(K_a^J E_J^b - \delta_b^a K_c^J E_J^c) && (28)\\
H &= \frac{1}{4\sqrt{det(q)}}(K_a^J K_b^L - K_a^L K_b^J)E_L^a E_J^b - \sqrt{det(q)}R, && (29)
\end{aligned}
$$

where $q_{ab} = \frac{E_a^I E_b^I}{det(q)}$, $det(q) = \sqrt{det(E_a^I)}$ and thus R is function of E_a^I.
We can equip our phase space with a symplectic structure where the (non-smeared) Poisson brackets are

$$
\begin{aligned}
\{E_I^a(x), E_J^b(y)\} &= 0 && (30)\\
\{K_a^I(x), K_b^J(y)\} &= 0 && (31)\\
\{E_I^a(x), K_b^J(y)\} &= \kappa\delta_I^J\delta_a^b\delta(x,y). && (32)
\end{aligned}
$$

We claim that the Hamiltonian system obtained via the new variables is equivalent to the ADM theory on the constraint surface $G_{IJ} = 0$. This means that if we propose this constraint then the functions (27) and (29) reduce to the corresponding functions on the ADM phase space, further more their Poisson brackets reduce to the ADM Poisson brackets [12].
Now we produce a finale transformation that will $a_)$ simplify the constraints and $b_)$ turn the rotation constraint G_{IJ} that corresponds to the Gauss constraint of a gauge theory. In other words we want to define an A_a^I SU(2) connection with the property

$$G_{JK} := (\partial_a E^a + [A_a, E^a])_{JK} = 0. \quad (33)$$

Before proceeding we need to make two important observations. First, for an arbitrary $\beta \neq 0$ complex number the rescaling $(K, E) \to (K(\beta) = \beta K, E(\beta) = E/\beta)$ is a canonical transformation. Here β is called the *Immirzi-parameter* and the interpretation of this rescaling is still an enigma in loop quantum gravity [14]. Second, it is obvious that $D_a e_b^I = 0$ and hence $D_a E_b^I = 0$.

With the use of it the rotational constraint becomes equivalent to

$$\begin{aligned} G_J &= \epsilon_{JKL}\beta K_a^K \frac{E_L^a}{\beta} = \epsilon_{JKL}\beta K_a^K \frac{E_L^a}{\beta} + D_a \frac{E_J^a}{\beta} = \\ &= \partial_a \frac{E_J^a}{\beta} + \epsilon_{JKL}(\Gamma_a^K + \beta K_a^K)\frac{E_L^a}{\beta} = \partial_a \frac{E_J^a}{\beta} + \epsilon_{JKL} A_a^K(\beta)\frac{E_L^a}{\beta}. \end{aligned} \tag{34}$$

The connection $A_a^K(\beta)$ could be called the Sen-Ashtekar-Immirzi-Barbero connection (names in historical order). The β dependence of $A_a^K(\beta)$ follows from the structure of Γ_a^K: it is easy to prove that Γ_a^K is independent of the value of β, thus $A_a^J(\beta) = \Gamma_a^J + \beta K_a^J$.
Since the Poisson brackets of A_a^I and E_J^b are

$$\{E_J^a(x), E_K^b(y)\} = 0 \tag{35}$$

$$\{A_a^J(x), A_b^K(y)\} = 0 \tag{36}$$

$$\{E_J^a(x), A_b^K(y)\} = \kappa\delta_J^K\delta_a^b\delta(x,y), \tag{37}$$

the pair (E_j^a, A_b^k) is a canonically conjugate pair, which can be explained by the fact that Γ_a^J depends only on E_a^J and its partial derivative. The fact that the Poisson bracket structure remains simple is one of the key ingredients to quantize the Hamiltonian theory of gravity. If it would be complicated then it would me a much harder task finding a suitable Hilbert space to represent A and E as well-defined operators.
What remains is to rewrite the Hamiltonian- and diffeomorphism constraints in terms of the new variables. Let us introduce the curvatures as

$$R_{ab}^J := 2\partial_{[a}\Gamma_{b]}^J + \epsilon_{JKL}\Gamma_a^K\Gamma_b^L \tag{38}$$

$$F_{ab}^J := 2\partial_{[a}A_{b]}^J + \epsilon_{JKL}A_a^K A_b^L \tag{39}$$

which are related with the covariant derivative via the $[D_a, D_b]v_J = \epsilon_{JKL}R_{ab}^K v^L$ and $[D_a, D_b]v_J = \epsilon_{JKL}F_{ab}^K v^L$ commutation rules. If we expand F in terms of Γ and K we get

$$F_{ab}^J = R_{ab}^J + 2\beta D_{[a}K_{b]}^J + \beta^2\epsilon_{JKL}K_a^K K_b^L.$$

After contracting it with E yields

$$F^J_{ab}E^b_J = \frac{R^J_{ab}E^b_J}{\beta} + 2D_{[a}K^J_{b]}E^b_J + \beta K^J_a G_J.$$

The first term on the right hand side of this equation vanishes identically because of the Bianchi identity ($\epsilon_{IJK}\epsilon^{cde}R^J_{cd}e^K_e = 0 \to \frac{1}{2}\epsilon_{IJK}\epsilon^{cde}R^J_{cd}e^K_e e^I_a = \frac{1}{2}E^b_J\epsilon_{cab}\epsilon^{cfe}R^J_{ae} = R^J_{ab}E^b_J$). If we compare this with the diffeomorphism constraint we obtain

$$F^J_{ab}E^b_J = H_a + \beta K^J_a G_J \tag{40}$$

Now we contract F^J_{ab} with $\epsilon_{JKL}E^a_K E^b_L$ to find

$$\begin{aligned} F^J_{ab}\epsilon_{JKL}E^a_K E^b_L &= det(q)\frac{R^{KL}_{ab}e^a_K e^b_L}{\beta^2} - 2\frac{E^a_J D_a G_J}{\beta} + \\ &+ (K^J_a E^a_J)^2 - (K^J_b E^a_J)(K^K_a E^b_K) \end{aligned} \tag{41}$$

For an arbitrary $v_a = e^I_a v_I$ the comparison of $[D_a, D_b]v_c$ and $[D_a, D_b]v_I$ leads to $R^{IJ}_{ab} = R_{abcd}e^a_I e^b_J$ and hence the first term in the above expression equals to $-det(q)\frac{R}{\beta^2}$. If we substitute it in the ADM Hamiltonian constraint we get

$$H = [F^J_{ab}(\beta) + (\beta^2 + \frac{1}{4})\epsilon_{JKL}K^K_a(\beta)K^L_b(\beta)]\frac{\epsilon_{JNM}E^a_N(\beta)E^b_M(\beta)}{\sqrt{det(q(\beta))}}. \tag{42}$$

To conclude let us write the constraints in the new variables as

$$G_J = D_a(\beta)E^a_J(\beta) \tag{43}$$

$$H_a = F^J_{ab}(\beta)E^b_J(\beta) \tag{44}$$

$$H = [F^J_{ab}(\beta) + (\beta^2 + \frac{1}{4})\epsilon_{JKL}K^K_a(\beta)K^L_b(\beta)]\frac{\epsilon_{JNM}E^a_N(\beta)E^b_M(\beta)}{\sqrt{det(q(\beta))}}. \tag{45}$$

We can see that the constraints in this form are simpler then in the ADM case. The Gauss-constraint is linear in both E and A, the diffeomorphism constraint is linear in E and quadratic in A, and if $\beta = \pm\frac{i}{2}$ then the scalar constraint is quadratic in both variables. This was the motivation of Ashtekar, to use the freedom in the Immirzi parameter to simplify the constraints. However, if one tries to quantize the theory he has to invoke reality conditions (since physical quantities have to be real), specifically one has to solve the quantized equations arising from

$$\frac{E(\beta)}{\beta} = \overline{\frac{E(\beta)}{\beta}} \tag{46}$$

$$\frac{A(\beta)-\Gamma}{\beta} = \overline{\frac{A(\beta)-\Gamma}{\beta}}, \tag{47}$$

where $\Gamma(E(\beta))$ is non-linear, not even analytic function of E^I_a. These conditions guarantee that there is no doubling in the number of degrees of freedom. In summary simplification of the scalar constraint implies reality conditions which are hard to implement at the quantum level, but if we work with real valued β then the constraints remain complicated. However recent researches have shown that one should use real Immirzi-parameters because the difficulties arising in the investigation of constraints can be solved, as it will be shown in section 4. From now on we will treat general relativity as a SU(2) gauge theory of gravity.

2.3 Classical theory of Matter fields

Since we will be investigating theories that have matter fields coupled to gravity, it is convenient to introduce the Lagrangian of these fields first and evaluate the 3+1 decomposition. Because our intention will focus on the scalar- and vector fields, we will not deal with fermionic fields here.

The Lagrangian density of the electromagnetic (U(1) Yang-Mills) and the U(1) scalar field coupled to gravity is

$$\mathcal{L} = \sqrt{-g}[g^{\mu\alpha}g^{\nu\beta}(-\frac{1}{4}\underline{F}_{\mu\nu}\underline{F}_{\alpha\beta}) - \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - U(\Phi)],$$

where $U(\Phi)$ denotes an arbitrary potential of the scalar field. To avoid confusion with the variables of the gravitational field we will denote the quantities of the electromagnetic field with an underline. Using the notations defined in the previous chapter we obtain

$$\begin{aligned}
\mathcal{L} &= N\sqrt{q}\Big[-\frac{1}{4}(q^{\mu\rho} - \frac{t^\mu - N^\mu}{N}\frac{t^\rho - N^\rho}{N})(q^{\nu\sigma} - \frac{t^\nu - N^\nu}{N}\frac{t^\sigma - N^\sigma}{N})\underline{F}_{\mu\nu}\underline{F}_{\rho\sigma} \\
&- \frac{1}{2}(q^{\mu\nu} - \frac{t^\mu - N^\mu}{N}\frac{t^\nu - N^\nu}{N})\partial_\mu\Phi\partial_\nu\Phi - U(\Phi)\Big] = \\
&= N\sqrt{q}\Big[-\frac{1}{4}q^{\mu\rho}q^{\nu\sigma}\underline{F}_{\mu\nu}\underline{F}_{\rho\sigma} + \frac{1}{2N^2}(\mathcal{L}_t\underline{A}_\rho - \mathcal{D}_\rho(\underline{A}_\nu t^\nu) - N^\nu\underline{F}_{\rho\nu})^2 -
\end{aligned}$$

$$- \quad \frac{1}{2}q^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2N^2}(\mathcal{L}_t\Phi - N^\nu\partial_\nu\Phi)^2 - U(\Phi)\Big]$$

Let us define the following quantities:

$$\begin{aligned}
\underline{A}_\mu t^\mu &= \underline{A}_0 \ , \quad \mathcal{D}_\mu = q^\nu_\mu \mathcal{D}_\nu \ , \quad \underline{A}_\mu = q^\nu_\mu \underline{A}_\nu \ , \quad \underline{B}^\mu = \frac{\sqrt{q}}{2}\epsilon^{\mu\nu\rho}q^\sigma_\nu q^\tau_\nu \underline{F}_{\sigma\tau} \ , \\
\underline{E}_\mu &= \frac{\sqrt{q}}{N}(\mathcal{L}_t\underline{A}_\mu - \mathcal{D}_\mu(\underline{A}_\nu t^\nu) - \epsilon_{\mu\nu\rho}\underline{B}^\nu N^\rho).
\end{aligned}$$

Since these quantities are three dimensional in the sense it was explained in the previous chapter, we shall use latin indices instead of the greek ones. After substituting them we obtain

$$\mathcal{L} = \frac{N}{\sqrt{q}}q^{ab}\frac{-\underline{B}_a\underline{B}_b + \underline{E}_a\underline{E}_b}{2} + N\sqrt{q}\Big[-\frac{1}{2}q^{ab}\partial_a\Phi\partial_b\Phi + \frac{1}{2N^2}(\mathcal{L}_t\Phi - N^a\partial_a\Phi)^2 - U(\Phi)\Big]$$

While observing the Lagrangian one finds that the time derivatives of $N, N_a, \underline{A}_0$ do not appear in its expression. The meaning of the first two is that the electromagnetic field and the scalar field does not give contributions to the gravitational primary constraints. The absence of $\dot{\underline{A}}_0$ means that we have an extra primary constraint, namely

$$\underline{\pi}_{A_0} := \frac{\delta S}{\delta \dot{\underline{A}}_0} = 0$$

To calculate the Hamiltonian first we need the canonical momenta for the fields Φ and $\underline{A}_a$:

$$\begin{aligned}
\Pi &= \frac{\delta S}{\delta\dot{\Phi}} = \frac{\sqrt{q}}{N}(\mathcal{L}_t\Phi - N^a\partial_a\Phi) && (48)\\
\underline{\pi}_a &= \frac{\delta S}{\delta\dot{\underline{A}}^a} = \underline{E}_a && (49)
\end{aligned}$$

Now let us perform the Legendre transformation:

$$\begin{aligned}
\mathcal{H} &= (N\frac{\underline{E}_a}{\sqrt{q}} + \mathcal{D}_a(\underline{A}^0) + \epsilon_{abc}\underline{B}^b N^c)\underline{E}^a + (\frac{N\Pi}{\sqrt{q}} + N^a\partial_a\Phi)\Pi - \\
&- \frac{N}{\sqrt{q}}q^{ab}\frac{-\underline{B}_a\underline{B}_b + \underline{E}_a\underline{E}_b}{2} - N\sqrt{q}\Big[-\frac{1}{2}q^{ab}\partial_a\Phi\partial_b\Phi + \frac{1}{2q}\Pi^2 - U(\Phi)\Big] = \\
&= \frac{N}{\sqrt{q}}q^{ab}\frac{\underline{B}_a\underline{B}_b + \underline{E}_a\underline{E}_b}{2} + \frac{N\sqrt{q}}{2}q^{ab}\partial_a\Phi\partial_b\Phi + \frac{N}{\sqrt{q}}\frac{\Pi^2}{2} + N\sqrt{q}U(\Phi) + \\
&+ \epsilon_{abc}\underline{B}^b N^c\underline{E}^a + N^a\partial_a\Phi\Pi + \mathcal{D}_a(\underline{A}^0)\underline{E}^a, && (50)
\end{aligned}$$

where the variables $\underline{E}_a, \underline{A}^a$ and Φ, Π form a canonical pair in the following sense:

$$\begin{aligned}
\{\underline{A}_a(x,t), \underline{A}^b(x',t)\} &= \{\underline{E}_a(x,t), \underline{E}^b(x',t)\} = 0 \\
\{\underline{E}_a(x,t), \underline{A}^b(x',t)\} &= \delta_a^b \delta(x-x') \\
\{\Phi(x,t), \Phi(x',t)\} &= \{\Pi(x,t), \Pi(x',t)\} = 0 \\
\{\Phi(x,t), \Pi(x',t)\} &= \delta(x-x')
\end{aligned}$$

Again, this Hamiltonian is a linear combination of secondary constraints since e.g. $\dot{\underline{\pi}}_{A_0} = 0 = \{\int d^3x \mathcal{H}, \underline{\pi}_{A_0}\}$ and this is true also for N_a and N. Since N and N_a are the same quantities as in the gravitational case the terms appearing in (50) proportional to N and N_a give contributions to the gravitational scalar- and diffeomorphism constraint, respectively. The last term is proportional to $\underline{A}^0$ which means we have an extra constraint, generating U(1) symmetry, namely the electromagnetic Gauss constraint:

$$\mathcal{D}_a \underline{E}^a = 0. \tag{51}$$

Although this constraint has the same form as the gravitational Gauss constraint, it is independent of that. To conclude let us write the constraints of this theory:

$$\begin{aligned}
H &= H^{GR} + \frac{1}{\sqrt{q}} q^{ab} \frac{\underline{B}_a \underline{B}_b + \underline{E}_a \underline{E}_b}{2} + \frac{\sqrt{q}}{2} q^{ab} \partial_a \Phi \partial_b \Phi + \frac{1}{\sqrt{q}} \frac{\Pi^2}{2} + \sqrt{q} U(\Phi) \\
H_a &= H_a^{GR} + \epsilon_{abc} \underline{B}^b \underline{E}^c + \partial_a \Phi \Pi \\
G_J &= D_a E_J^a \\
\underline{G} &= \mathcal{D}_a \underline{E}^a
\end{aligned} \tag{52}$$

where H^{GR} and H_a^{GR} are contributions of the gravitational field.

3 Quantization

In this section we will introduce a method which will enable us to quantize the gravitational (and any other) field in a diffeomorphism covariant way. The method is called Refined Algebraic Quantization (RAQ) and it consists of the following steps:

- First one does the 3+1 decomposition to arrive to system with first class constraints - this is what we did in the previous section. The case where the constraints are not first class will be dealt with in the case of the Proca-field.

- One searches for representation of the basic variables to define corresponding operators on a kinematic Hilbert-space $\mathcal{H}_{kin}$ such that the adjointness relations and the Poisson brackets are implemented. The Hilbert-space is usually of the form $L_2(\overline{\mathcal{C}}, d\mu)$ where $\overline{\mathcal{C}}$ is some distributional extension of the configuration space $\mathcal{C}$ and $d\mu$ is a suitable measure.

- We need a representation which supports the constraints. This is not a trivial step if one - as in the present case - has constraints which are not linear.

- If the constraints are implemented we need to solve them. If C is a constraint then the natural solution would be those $f \in \mathcal{H}$ for which

  $$\hat{C}f = 0.$$

  But we need to generalize this since often the solutions are distributions which do not lie in $\mathcal{H}$. So one looks for distributions l such that

  $$l(\hat{C}f) = 0 \; \forall f \in \mathcal{H}.$$

This section is devoted to finding suitable operators and the construction of the Hilbert-space, while the implementation of the constraints and their solution will be dealt with in the next section.

3.1 Quantization of the gravitational field

Now what we have to do is the apply the previous points to general relativity. The first step has been made since the phase space $\mathcal{M}$ contains canonical conjugate pairs (A^a_μ, E^μ_a) where A is an SU(2) connection and E is a SU(2) vector field on σ, so we can define the configuration space $\mathcal{C}$ to be $\mathcal{A}$, the space of smooth SU(2) connections on σ. The following section will provide the basic ingredients to the other points, where we use especially the construction found in [12] and [15].

3.1.1 Quantizing the Ashtekar variables

To define the Hilbert-space one has to use the smeared version of the connection A^I_a since the functional derivatives of polonomyals in A^I_a are proportional to a delta distribution. A natural candidate would be of the form $F[A] := \int_\sigma F^a_I(x) A^I_a(x)$ but this does not transforms nicely under SU(2) transformation. Instead we will use the notion of the *holonomy*:

Definition 3.1 *Let $c : [0,1] \to \sigma$ be a curve on σ and $A \in \mathcal{A}$ a connection. The* *holonomy/parallel transport* *$A(c) := h_{c,A} \in SU(2)$ is the unique solution of the following differential equation:*

$$\frac{dh_{c_t}(A)}{dt} = h_{c_t}(A) A^I_a(c(t)) \frac{\tau_I}{2} \dot{c}^a(t) \tag{53}$$

$$h_{c_0}(A) = 1, \tag{54}$$

where τ_I are the generators of $SU(2)$.

It is easy to see that this definition is equivalent with

Definition 3.2

$$A(c) = \mathcal{P}(\exp(\int_c A)) = 1 + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_{t1}^1 dt_2 \ldots \int_{tn-1}^1 A(t_1) \ldots A(t_n), \tag{55}$$

where $\mathcal{P}$ denotes the path ordering symbol which orders the curve parameters from left to right according to their value beginning with the smallest one and $A(t) := A^I_a(c(t)) \dot{c}^a \tau_J/2$.

Two important properties of the holonomy (which can be derived from the definition) are

$$h_{e_1 \circ e_2}(A) = h_{e_1}(A) h_{e_2}(A)$$

$$h_{e^{-1}}(A) = h_e^{-1}(A)$$

Further more it follows from the definition that under gauge transformations the holonomy transforms as $h_c(A^g) = g(c(0))h_c(A)g(c(1))^{-1}$. It is also clear from this expression that gauge invariant objects can be constructed by closed curves (loops).
Let us turn to the case of the electric field. Since we want the commutator to be non-distributional, the electric field has to be smeared in at least two dimensions. But if we smear it in three dimensions with smearing function f then the Poisson bracket will be

$$\{E(f), h(A)_{mn}\} = \int_0^1 \dot{c}^a(t) f_a^J(t) [h_{c_t}(A) \frac{\tau_J}{2} h_{c_t}(A)^{-1}]_{mn}, \tag{56}$$

which is a continuous linear combination of the holonomies (because of the integral), which means that the algebra does not close. So the smearing has to be done in two dimensions, leading to the definition of the *electric flux*:

$$E_j(S) := \int_S *E_j \tag{57}$$
$$E(S) := E_j(S)\tau_j, \tag{58}$$

where $(*E)^j_{ab} := E^c_j \epsilon_{abc}$ and S is a open two dimensional surface. From the definition we can derive how the electric flux transforms under gauge transformations and diffeomorphism:

$$E^g(S) = \int_S Ad_g(*E) \tag{59}$$
$$E^\phi(S) = E(\phi^{-1}(S)), \tag{60}$$

where $Ad_g(*E) := g(*E)g^{-1}$. Another benefit of the previous definitions is that the transformation rule under spatial diffeomorphisms for these smeared quantities is simple. Using $V_a := H_a - A^j_a G_j$ (diffeomorphism modulo Gauss constraint) we have

$$\{\vec{V}(\vec{N}), h(A)\} = \mathcal{L}_{\vec{N}} h(A)$$
$$\{\vec{V}(\vec{N}), E(S)\} = \mathcal{L}_{\vec{N}} E(S)$$

All in all the electric flux and the holonomy are the best candidates defining multiplication and momentum operators. But before we do this we have to define the Hilbert space which they will act on.

3.1.2 Hilbert-space for the gravitational field

The main problem with the definition of the Hilbert-space is that in field theory we have infinite degrees of freedom. To solve this the construction is based upon the idea of Kolmogorov: first we start with finite degrees of freedom then extend it to infinite many degrees of freedom. To see how this is done we first introduce some definitions.

Definition 3.3 *A curve is a $c : [0,1] \to \sigma$ map which is continuous,oriented and piecewise semi-analytic. The beginning- and endpoints of the curve are denoted $b(c) := c(0)$ and $f(c) := c(1)$ respectively. The range of the curve is $r(c) := c([0,1])$. The set of curves is denoted by $\mathcal{C}$*

A piecewise semi-analytic curve differs form a piecewise analytic in the points of non-analyticity: here the latter has to be only continuous, but the former has to be C^1.

Definition 3.4 *The composition of composable curves (for which $f(c_1) = b(c_2)$) is defined as*

$$[c_1 \circ c_2](t) = \begin{cases} c_1(2t) & t \in [0, \frac{1}{2}] \\ c_2(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

while the inverse is $c^{-1}(t) = c(1-t)$.

It is useful to define an equivalence relation on the set of curves.

Definition 3.5 *Two c, c' curves are said to be equivalent $(c \sim c')$ if there exists a $f(t) : [0,1] \to [0,1]$ diffeomorphism such that $c(t) = c'(f(t))$ or c and c' can be obtained as $c = s_1 \circ s_2 \quad c' = s_1 \circ s_3 \circ s_3^{-1} \circ s_2$ (or finite combinations of these operations). The equivalence class of curves are called paths (p) while the set of paths are denoted by $\mathcal{P}$.*

The importance of this definition is that the holonomy depends only of the path, not the curve. That is if $(c \sim c')$ then

$$h_c(A) = h_{c'}(A) \tag{61}$$

The second advantage of dealing with paths instead of curves is that we almost have a group structure since composition is associative and the path $p_c \circ p_{c^{-1}} = b(c)$ is trivial. However we still do not have a natural identity and not all elements can be composed. The natural structure behind is a groupoid, which is a special category.

Definition 3.6 *A category $(\mathcal{K})$ is a class, the members of which are called objects $x, y, \ldots$ together with a collection $M(\mathcal{K})$ of sets $hom(x, y)$ for each ordered pair (x,y), the members of which are called morphisms. Between the sets of morphisms there is a composition operation*

$$\circ : \; hom(x, y) \times hom(y, z) \to hom(x, z); \quad (f, g) \mapsto f \circ g$$

which has to satisfy the following rules:
1) Associativity: $f \circ (g \circ h) = (f \circ g) \circ h$ for all $f \in hom(w, x), g \in hom(x, y), h \in hom(y, z)$
2) Identities: for every $x \in \mathcal{K}$ there exists a unique element $id_x \in hom(x, x)$ such that for all $y \in \mathcal{K}$ we have $id_x \circ f = f$ for all $f \in hom(y, x)$ and $f \circ id_x = f$ for all $f \in hom(x, y)$.
A subcategory $\mathcal{K}' \subset \mathcal{K}$ is a category which contains a subclass of the class of objects in $\mathcal{K}$ and and for each pair of objects (x,y) in $\mathcal{K}'$ we have for the sets of morphisms $hom'(x, y) \subset hom(x, y)$.
A morphism $f \in hom(x, y)$ is called an isomorphism provided there exists $g \in hom(y, x)$ such that $f \circ g = id_y$ and $g \circ f = id_x$.
A groupoid is a category where every morphism is an isomorphism.

This definition obviously applies to our situation with the following identifications:
Category: σ
Objects: points $x \in \sigma$
Morphisms: paths between points $hom(x, y) := \{p \in \mathcal{P}; b(p) = x; f(p) = y\}$. Obviously every morphism is a isomorphism.
Collection of sets of morphisms: all paths $M(\sigma) = \mathcal{P}$
Composition: composition of paths.
Identities: $id_x = p \circ p^{-1}$ for any $p \in \mathcal{P}$ with $b(P) = x$
The reason for us to arrive to this definition because this will be the main tool in "discretizing" σ. Also this groupoid structure will enable us to define states for the following reason: First the holonomy depends only on the path - to express this we will use the notation $A(p_c) := h_c(A);\ A \in \mathcal{A}$. Second $A(p_1 \circ p_2) = A(p_1)A(p_2), A(p^{-1}) = A(p)^{-1}$, thus every A defines a groupoid morphism. Thus we have the following definition:

Definition 3.7 *$Hom(\mathcal{P}, G)$ is the set of all groupoid morphisms from the set of paths in σ to the gauge group.*

In order to define the level of this discretization without introducing a metric, one uses projective techniques.

Definition 3.8 *Let $\mathcal{L}$ some abstract label (index) set. A partial order $\prec$ is a relation on $\mathcal{L}$, i.e. a subset of $\mathcal{L} \times \mathcal{L}$ which is symmetric, reflexive and transitive. Not all elements of $\mathcal{L}$ need to be in relation and if they are, $\mathcal{L}$ is said to be linearly ordered.*
A partially ordered set is said to be directed if for any $l, l' \in \mathcal{L}$ there exists l'' such that $l, l' \prec l''$.
Let $\mathcal{L}$ be a partially ordered, directed index set. A projective family $(X_l, p_{l'l})_{l \prec l' \in \mathcal{L}}$ consists of sets X_l labelled by $\mathcal{L}$ together with surjective projections

$$p_{l'l} : X_{l'} \to X_l \ \forall l \prec l'$$

satisfying the consistency condition

$$p_{l'l} \circ p_{l''l'} = p_{l''l} \ \forall l \prec l' \prec l''$$

The projective limit $\overline{X}$ of a projective family $(X_l, p_{l'l})$ is the subset of the direct product $X_\infty := \prod_{l \in \mathcal{L}} X_l$ defined by

$$\overline{X} := \{(x_l)_{l \in \mathcal{L}}; p_{l'l}(x_{l'} = x_l \ \forall l \prec l')\}$$

To identify the label set and the projective family in our case we define the notion of edge and graph.

Definition 3.9
1) *An edge is a path which is semi-analytic in all $[0, 1]$*
2) *A finite set of edges $\{e_1, \ldots, e_N\}$ is said to be independent provided that e_k intersect each other at most in their endpoints $b(e_k), f(e_k)$.*
3) *A finite set of edges $\{e_1, \ldots, e_N\}$ is said to be algebraically independent provided that none of the e_k is a finite combination of the other edges and their inverses.*
4) *An independent set of edges defines an oriented graph γ by $\gamma := \cup_{n=1}^N r(e_k)$ where $r(e_k)$ carries the arrow induced by e_k. From γ we can recover the its set of edges $E(\gamma) := \{e_1, \ldots, e_N\}$*

as the maximal semi-analytic segments of γ together with their orientations as well as the set of vertices $V(\gamma) := \{b(e), f(e), e \in E(\gamma)\}$. The set of all finite, semi-analytic graphs is denoted by Γ_0^ω.

5) *Given a graph we denote by $l(\gamma) \subset \mathcal{P}$ the subgroupoid generated by γ with $V(\gamma)$ as the set of objects and with the $e \in E(\gamma)$ together with their inverses and finite compositions as the set of homomorphisms.*

The labels ω and 0 stand for 'semi-analytic' and 'compact support' respectively. The following theorem will explain why we introduced this definition.

Theorem 3.1 *Let $\mathcal{L}$ be the set of all tame subgroupoids $l(\gamma) \in \mathcal{P}$, that is, those determined by graphs $\gamma \in \Gamma_0^\omega$. Then the relation $l \prec l'$ - meaning l is a subgroupoid of l' - equips $\mathcal{L}$ with the structure of a partially ordered and directed set.*

Now that we have a partially ordered and directed set we can define a projective family.

Definition 3.10

1) *For any l define $X_l := Hom(l, G)$, the set of all homomorphisms from the subgroupoid l to G.*

2) *For $l \prec l'$ define a projection by $p_{l'l} : X_{l'} \to X_l$ the restriction of the homomorphism defined on the subgroupoid l' to its subgroupoid l.*

Applying the properties of graphs and the projections, one can construct the projective limit and prove the following theorem [12]:

Theorem 3.2 *The projective limit $\overline{X}$ of the spaces $X_l := Hom(l, G)$, $l \in \mathcal{L}$ - where $\mathcal{L}$ denotes the set of all tame subgroupoids of $\mathcal{P}$ - is a compact Hausdorff space in the induced Tychonoff topology whenever G is a compact Hausdorff topological group.*

The crucial point of the above theorem is the compactness of G which is why one has to make the 3+1 decomposition to use the tools derived here (otherwise we would have to deal with non-compact groups).

The main importance of this theory is that we may identify $\bar{\mathcal{A}} := Hom(\mathcal{P}, G)$ with $\overline{X}$ so we can deal with the continuous degrees of freedom with finite degrees of freedom. The strategy will be to define quantum theory on a general X_l and extend it to $\overline{X}$.

We cannot say yet that $\overline{X}$ is our Hilbert-space since we also need a scalar product. To define this we first need the notion of so-called cylindrical functions. The general definition of cylindrical functions - for an abstract partially ordered and directed set $\mathcal{L}$ which labels compact Hausdorff spaces X_l with surjective and continuous projections $p_{l'l}$ which satisfy the consistency conditions - is the following:

Definition 3.11
1) Let $C(X_l)$ be the continuous, complex valued functions on X_l and consider their union:

$$Cyl'(\overline{X}) := \cup_{l \in \mathcal{L}} C(X_l)$$

Given $f, f' \in Cyl'(\overline{X})$ we find l and l' such that $f \in C(X_l), f' \in C(X_{l'})$ and we say that f and f' are equivalent ($f \sim f'$) provided that

$$p^*_{l''l} f = p^*_{l''l'} f' \quad \forall l, l' \prec l''$$

(pull-back maps).
2) The space of cylindrical functions on the projective limit $\overline{X}$ is defined to be the equivalence classes

$$Cyl(\overline{X}) := Cyl'(\overline{X}) / \sim .$$

The equivalence class of $f \in Cyl'(\overline{X})$ will be denoted $[f]_\sim$.

Before we carry on to the definition of the measure, we will stress out some of the important properties of $Cyl(\overline{X})$.

Lemma 3.1 *Given $f, f' \in Cyl(\overline{X})$ there exists a common label $l \in \mathcal{L}$ and $f_l, f'_l \in C(X_l)$ such that $f = [f_l]_\sim, f' = [f'_l]_\sim$*

Lemma 3.2
1) Let $f, f' \in Cyl(\overline{X})$. Then the following operations are well defined:

$$f + f' := [f_l + f'_l]_\sim, f f' := [f_l f'_l]_\sim, z f := [z f_l]_\sim, \bar{f} := [\bar{f}_l]_\sim,$$

where $z \in \mathbb{C}$ and $\bar{f}_l$ means complex conjugation.
2) $Cyl(\overline{X})$ contains the constant function.

3) The sup-norm for $f = [f_l]_\sim$ is $||f||$ and is well defined:

$$||f|| := \sup_{x_l \in X_l} |f_l(x_l)|$$

This lemma tells us that $Cyl(\overline{X})$ is a normed space and a unital, Abelian *-algebra. Because of this $Cyl(\overline{X})$ is also a metric space ($d(f, f') = ||f - f'||$), which can be uniquely embedded into a complete metric space. If we complete $Cyl(\overline{X})$ in the sup-norm, we obtain an Abelian, unital Banach *-algebra $\overline{Cyl(\overline{X})}$. In addition one can prove that the C^* property $||f\bar{f}|| = ||f||^2$ also holds, thus this algebra is a C^*-algebra. This suggests that we should use some of the results provided by Gel'fand- Naimark-Segal theory.

Let $\Delta(\overline{Cyl(\overline{X})})$ be the spectrum of $\overline{Cyl(\overline{X})}$, that is the set of *all* homomorphisms from $\overline{Cyl(\overline{X})}$ to the complex numbers and denote the Gel'fand isometric isomorphism by

$$\bigvee : \overline{Cyl(\overline{X})} \to C(\Delta(\overline{Cyl(\overline{X})})); \; f \mapsto \check{f}; \; \check{f}(\chi) := \chi(f)$$

where the space of continuous functions on the spectrum is equipped with the sup-norm. Notice the similarities between the spaces $\overline{Cyl(\overline{X})}$ and $C(\Delta(\overline{Cyl(\overline{X})}))$: both are spaces of continuous functions over compact Hausdorff spaces and on both spaces the norm is sup-norm. This suggests that there is a homeomorphism between $C(\overline{X})$ and $Hom(\overline{Cyl(\overline{X})}, \mathbb{C})$. This follows from the following theorem.

Theorem 3.3 *The map $\chi : \overline{X} \to C(\overline{Cyl(\overline{X})})$; $x = (x_l)_{l \in \mathcal{L}} \mapsto \chi(x)$ where $[\chi(x)](f) := f_l(p_l(x))$ for $f = [f_l]_\sim$ is a homeomorphism.*

To summarize, *the closure of the space of cylindrical functions $\overline{Cyl(\overline{X})}$ may be identified with the space of continuous functions $C(\overline{X})$ on the projective limit $\overline{X}$.*

Now we need a suitable measure. The tricky part is that this measure is not defined directly on $\overline{X}$ but this is induced by a family of measures μ_l defined on each X_l. Of course this family of measures has to be consistent in some way.

Definition 3.12 *A family of measures $(\mu_l)_{l \in \mathcal{L}}$ on the projections X_l of a projective family $(X_l, p_{l'l})_{l \prec l' \in \mathcal{L}}$ where the projections $p_{l'l} : X_{l'} \to X_l$ are continuous and surjective is said to be <u>consistent</u> provided that*

$$(p_{l'l})_* \mu_{l'} := \mu_{l'} \circ p_{l'l}^{-1} = \mu_l$$

for any $l \prec l'$. The measure $(p_{l'l})_*\mu_{l'}$ on X_l is called the push-forward of the measure $\mu_{l'}$.

The consistency condition in def. 3.12 can be rewritten in the following way:

$$\int_{X_{l'}} d\mu_{l'}(x_{l'})\chi_{p_{l'l}^{-1}(U_l)}(x_{l'}) = \int_{X_l} d\mu_l(x_l)\chi_{U_l}(x_l),$$

where χ_S denotes the characteristic function of a set S. In this from we may interpret the consistency condition as integrating out the degrees of freedom in $X_{l'}$ on which $p_{l'l}^* f_l$ does not depend. It follows that if $f = [f_l]_\sim \in Cyl(\overline{X})$ with $f_l \in C(X_l)$ then the linear functional

$$\Lambda : Cyl(\overline{X}) \to \mathbb{C};\ f = [f_l]_\sim \mapsto \Lambda[f] := \int_{X_l} d\mu_l(x_l) f_l(x_l)$$

is well defined. This property is important to prove the following theorem:

Theorem 3.4 *Let $(X_l.p_{l'l})_{l\prec l' \in \mathcal{L}}$ be a compact Hausdorff projective family with continuous and surjective projections $p_{l'l} : X_{l'} \to X_l$, projective limit $\overline{X}$ and projections $p_l : \overline{X} \to X_l$.*
1) If μ is a regular Borel probability measure on $\overline{X}$ then $(\mu_l := \mu \circ p_l^{-1})_{l\in\mathcal{L}}$ defines a consistent family of regular Borel probability measures on X_l.
2) If $(\mu_l)_{l\in\mathcal{L}}$ defines a consistent family of regular Borel probability measures on X_l then there exists a unique, regular Borel probability measure μ on $\overline{X}$ such that $\mu \circ p_l^{-1} = \mu_l$.
3) The measure μ is faithful if and only if every μ_l is faithful.

So the method is simple: we define faithful regular Borel measures μ_l on X_l, which ensure that a unique μ on $\overline{X}$ exists. Note that we do not necessarily need to derive the exact form of μ; we may work on a general X_l space and generalize the result.
The last ingredient is to define the measure on $\overline{\mathcal{A}}$. To do this we must specify the space of cylindrical functions. Given a subgroupoid $l \in \mathcal{L};\ l = l(\gamma)$ we think of an element $x_l \in X_l$ as a collection of group elements $\{x_l(e)\}_{e\in E(\gamma)} = p_l(x_l)$ and X_l can be identified with $G^{|E(\gamma)|}$. Thus an element $f_l \in C(X_l)$ is simply given by

$$f_l(x_l) = F_l(\{x_l(e)\}_{e\in E(\gamma)}) = (p_l^* F_l)(x_l)$$

where F_l is a complex valued function on $G^{|E(\gamma)|}$. Exploiting the fact that G is compact we define our measure with the help of the Haar measure. To be precise the measure is defined as follows:

Definition 3.13 *Let $\mathcal{L}$ be the set of all subgroupoids of the set of semi-analytic paths $\mathcal{P}$ in σ and $X_l = Hom(l, G)$ identified with $G^{|E(\gamma)|}$ if $l = l(\gamma)$. Then for any $f \in C(X_l)$ we define*

$$\mu_{0l}(f_l) = \int_{X_l} d\mu_{0l}(x_l) p_l^* F_l(x_l) := \int_{G^{|E(\gamma)|}} \left[\prod_{e \in E(\gamma)} d\mu_H(h_e) \right] F_l(\{h_e\}_{e \in E(\gamma)})$$

where μ_H is the Haar probability measure on G.

Since G is compact the Haar measure is invariant under left and right translations and under inversions. This is important because these are required to prove that the family of measures μ_{0l} defines a regular Borel probability measure on $\overline{X}$. Thus we may introduce our Hilbert-space.

Definition 3.14 *The Hilbert-space $\mathcal{H}_0$ is defined as the space of square integrable functions over $\overline{\mathcal{A}}$ withe respect to the uniform measure μ_0, that is*

$$\mathcal{H} := L_2(\overline{\mathcal{A}}, \mu_0)$$

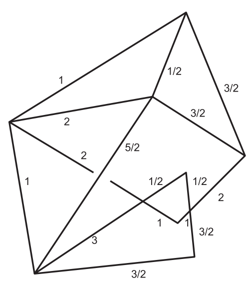

Figure 2: A spin network.

An important concept in Loop Quantum Gravity is the so-called *spin-network function* [24],[25] because they give a *complete orthonormal basis* on our Hilbert-space.

Definition 3.15 *Fix once and for all a representative from each equivalence class of irreducible representations of the compact Lie group G and denote the collections of these representatives by Π. Let $l = l(\gamma)$ be given and associate with every edge $e \in E(\gamma)$ a non-trivial, irreducible representation $\pi_e \in \Pi$ which we assemble in a vector $\vec{\pi} := (\pi_e)_{e\in E(\gamma)}$. Consider for each vertex v_j the the set of contractors I_j that are intertwining operators (or just simply intertwiners) mapping from the tensor product of the representations of the incoming edges to the tensor product of the representations of the outgoing edges. A spin network state is a cylindrical function of the following form:*

$$T_{\gamma,\vec{\pi},\vec{I}} := \otimes_{i=1}^{|E(\gamma)|} \pi(h_{e_i}(A)) \cdot \otimes_{j=1}^{|V(\gamma)| I_j}, \tag{62}$$

where $\cdot$ means contracting the the upper indices of the intertwiner with all of the incoming edges and the lower indices with all of the outgoing edges.

3.1.3 Regularisation of the holonomy-flux Poisson algebra

After introducing the elementary operators we need their algebra and action on any element of the Hilbert-space to continue. The two problems are connected since - as we shall see later - the action of the operators will be based upon the commutator algebra. The main problem is that our operators are based on quantities that are smeared in one and two dimensions respectively, while the Poisson bracket of the Ashtekar variables are well-defined if both quantities are smeared in three dimensions. That is why one has to regulate the Poisson bracket. Though this process is quite lengthy and technical [36], we will introduce it in detail since the methods used there are very common in Loop Quantum Gravity (we will also use them in case of the constraint operators).

Regularisation

The strategy will be to regularize the functions $A(p), E(S)$ in order to arrive at a 3-dimensional smearing, then to compute the Poisson brackets of the regulated functions and finally we will remove the regulator and hope to arrive at a well-defined symplectic structure.

First we define a *tube* T_p^ϵ with central path p to be a smooth function of the form

$$T_p^{\epsilon t} : \mathbb{R}^2 \times [0,1] \to \sigma;\ T_p^{\epsilon t}(s_1, s_2, t') := \delta^\epsilon(t-t')\delta^\epsilon(s_1, s_2) p_{s_1,s_2}(t'),$$

where $p_{s_1,s_2}(t')$ is a smooth assignment of mutually non-intersecting paths diffeomorphic to $p := p_{0,0}$ and δ^ϵ is a smooth regularisation of the delta distribution. We then define (recall the definition of the holonomy)

$$h_p^\epsilon(A) := \mathcal{P}\exp(\int_{\mathbb{R}^2} d^2 s \delta^\epsilon(s_1, s_2) \int_0^1 dt \int_{p_{s_1,s_2}} dt' \delta_t^\epsilon A),$$

where path ordering is with respect to the t parameter. We obviously have $\lim_{\epsilon\to 0} h_p^\epsilon(A) = h_p(A)$ point-wise in $\mathcal{A}$ for every choice of δ^ϵ.

Likewise we define a disk D_S^ϵ with central surface S to be a smooth function of the form

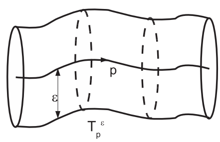

Figure 3: Regularisation of a holonomy with a tube.

$$D_S^\epsilon : \mathbb{R} \times U \to \sigma;\ D_S^\epsilon(p, s_1, s_2) := \delta^\epsilon(p) S_p(s_1, s_2),$$

where S_p is a smooth assignment of mutually non-intersecting surfaces diffeomorphic to $S := S_0$ and U denotes the subset of $\mathbb{R}^2$ in the pre-image of S. We then define

$$E_n^\epsilon(S) := \int_{\mathbb{R}} dp\delta^\epsilon(p)E_n(S_p).$$

We obviously have $\lim_{\epsilon\to 0} E_n^\epsilon(S) = E_n(S)$ point-wise in $\mathcal{E}$, the space of smooth electric fields over σ.

Recall the three dimensionally smeared Ashtekar connection and electric field: $A(F) := \int d^3xA_j^aF_a^j,\ E(f) := \int d^3xE_a^jf_j^a$. Since the holonomy and electric flux are smeared in one and two dimensions respectively we introduce the following regulators:

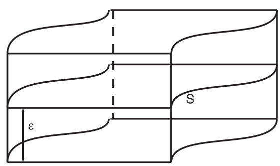

Figure 4: Regularisation the electric flux with a disk.

$$
\begin{aligned}
F_p^{\epsilon k t}(x)_j^a &:= \delta_j^k \int_{\mathbb{R}^2} d^2 s \delta^\epsilon(s_1, s_2) \times \\
&\times \int_0^1 dt' \delta(t'-t) \dot{p}^a_{s_1,s_2}(t') \delta(x, p_{s_1,s_2}(t')) \\
f_S^{\epsilon n}(x)_a^j &:= n^j(x) \int_{\mathbb{R}} ds \delta^\epsilon(s) \int_U d^2 u \epsilon_{a a_1 a_2} \times \\
&\times \frac{\partial S_s^{a_1}(u_1,u_2)}{\partial u_1} \frac{\partial S_s^{a_2}(u_1,u_2)}{\partial u_2} \delta(x, S_s(u_1,u_2))
\end{aligned}
\tag{63}
$$

It is easy to see that

$$
\begin{aligned}
h_p^\epsilon(A) &= \mathcal{P}\exp(\int_0^1 dt F_p^{\epsilon j t}(A) \tau_j/2) \\
E_n^\epsilon(S) &= E(f_S^{\epsilon n})
\end{aligned}
\tag{64}
$$

To calculate the Poisson bracket between $E_n^{\epsilon'}(S)$ and $h_e^\epsilon(A)$ one first carefully expends the path-ordered exponential and uses the Leibniz rule:

$$
\begin{aligned}
\{E_n^{\epsilon'}(S), h_e^\epsilon(A)\} &= \sum_{n=1}^\infty \int_0^1 dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1 \times \\
&\times \sum_{k=1}^n (F_e^{\epsilon j_1 t}(A)\tau_{j_1}/2) \ldots (F_e^{\epsilon j_{k-1} t}(A)\tau_{j_{k-1}}/2) \times \\
&\times \{E(f_S^{\epsilon' n}), F_e^{\epsilon j_k t_k}(A)\}\tau_k/2 (F_e^{\epsilon j_{k+1} t}(A)\tau_{j_{k+1}}/2) \ldots (F_e^{\epsilon j_n t}(A)\tau_{j_n}/2).
\end{aligned}
$$

Using

$$
\begin{aligned}
\{E(f_S^{\epsilon' n}), F_e^{\epsilon k t}(A)\} &= \int_{\mathbb{R}^2} d^2 s \delta^\epsilon(s_1,s_2) \int_{\mathbb{R}} ds \delta^{\epsilon'}(s) \int_0^1 dt' \delta^\epsilon(t'-t) \int_U d^2 u \dot{e}^a_{s_1,s_2}(t') \epsilon_{a a_1 a_2} \times \\
&\times \frac{\partial S_s^{a_1}(u_1,u_2)}{\partial u_1} \frac{\partial S_s^{a_2}(u_1,u_2)}{\partial u_2} \times \delta(S_s(u_1,u_2), e_{s_1,s_2}(t')) n^k(S_s(u_1,u_2))
\end{aligned}
$$

we take *first* the limit $\epsilon \to 0$ and *then* $\epsilon' \to 0$ (the reason for doing this will become transparent later). The result is

$$
\{E_n^{\epsilon'}(S), h_e^\epsilon(A)\} = \sum_{n=1}^\infty \int_0^1 dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1 \times
$$

$$\times \sum_{k=1}^{n} A(t_1)\ldots A(t_{k-1}) \lim_{\epsilon\to 0}[\{E(f_S^{\epsilon' n}), F_e^{\epsilon j_k t_k}(A)\}]\tau_k/2A(t_{k+1})\ldots A(t_n),$$

where $A(t) := A_a^j(e(t))\dot{e}^a\tau_j/2$ and the limit of the square bracket is

$$\int_{\mathbb{R}} ds\delta^{\epsilon'}(s) \int_U d^2u\dot{e}^a(t_k)n^{j_k}(p(t_k))\epsilon_{aa_1a_2} \times \\ \times \frac{\partial S_s^{a_1}(u_1,u_2)}{\partial u_1}\frac{\partial S_s^{a_2}(u_1,u_2)}{\partial u_2}\delta(S_s(u_1,u_2),p(t_k)) \tag{65}$$

Fortunately there is an additional t_k integral involved so the end result will be non-distributional. Now let $t \mapsto F(t)$ be any integrable function and consider the integral

$$\int_{\mathbb{R}} ds\delta^{\epsilon'}(s) \int_U d^2u \int_0^{t_{k-1}} dtF(t)\dot{e}^a(t)\epsilon_{aa_1a_2} \times \\ \times \frac{\partial S_s^{a_1}(u_1,u_2)}{\partial u_1}\frac{\partial S_s^{a_2}(u_1,u_2)}{\partial u_2}\delta(S_s(u_1,u_2),e(t)) \tag{66}$$

The appearance of the delta distribution $\delta(S_s(u_1,u_2),e(t))$ forces us to discuss this limit depending on how the edge is oriented to the surface. It is easy to see that there are four cases:

1) *up*

Here $e \cap S = b(e)$ is an isolated intersection point and the beginning segment of e lies in U_+ (we have oriented surfaces).

2) *down*

Here $e \cap S = b(e)$ is an isolated intersection point and the beginning segment of e lies in U_-.

3) *inside*

Here $e \cap \bar{S} = e$, that is e lies entirely in S.

4) *outside*

Here $e \cap S = 0$, that is e lies outside of S.

If we look at the structure of our Hilbert-space one can conclude that each graph can be transformed into a graph which is a) equivalent with the original graph and b) all of its edges are one of the previous types. Now let us calculate the integral.

Case outside

This case is trivial since for small ϵ' the delta distribution vanishes identically.

Case inside

It is clear that in this case the function $\delta(S_s(u_1,u_2),e(t))$ has support at $s=0$ and when u_1,u_2 are the unique solutions of the equation $S_0(u)=e(t)$. Thus the integral becomes

$$\delta^{\epsilon'}(0)\int_0^{t_{k+1}} dtF(t)\frac{\dot{e}^a(t)\epsilon_{aa_1a_2}\left[\frac{\partial S_s^{a_1}(u_1,u_2)}{\partial u_1}\frac{\partial S_s^{a_2}(u_1,u_2)}{\partial u_2}\right]_{u(t)}}{det(\partial S_s(u_1,u_2)/\partial(s,u_1,u_2))_{s=0,u=u(t)}} \tag{67}$$

which vanishes for *finite* ϵ' since the denominator is finite while the numerator vanishes by definition of an inside edge (this is everywhere tangential to the surface). Since for every finite ϵ' (67) vanishes, its limit for $\epsilon'\to 0$ also vanishes. This is the reason why we did not synchronize the limits of ϵ' and ϵ ; in that case we would have obtained an ill defined limit $0\cdot\infty$.

Case up

For sufficiently small ϵ' and every $s>0$ the edge e cuts the surface transversally in a single interior point $q_s=e(t_s)=S_s(u_s)$. The outward normal direction in this case is

$$n_a^s(u):=\epsilon_{aa_1a_2}\frac{\partial S_s^{a_1}(u_1,u_2)}{\partial u_1}\frac{\partial S_s^{a_2}(u_1,u_2)}{\partial u_2}$$

and one can show that for $s>0$ the combination $\dot{e}^a(t_s)n_a^s(u_s)$ is positive. If $s=0$ this might vanish but in this case the point $s=0$ is of zero measure. Thus we may evaluate the integral by changing to new coordinates $X_s(t,u)=S_s(u)-e(t)$, the Jacobean of which is $|\dot{e}^a(t)n_a^s(u)|$. After evaluating the delta distribution $\delta(X_s(t,u))$ we obtain

$$\int_{\mathbb{R}} ds\delta^{\epsilon'}(s)\theta(t_{k+1}-t_s)\theta(s)F(t_s)\frac{\dot{e}^a(t_s)n_a^s(u_s)}{|\dot{e}^a(t)n_a^s(u)|} \tag{68}$$

where $\theta(x)$ is the step function. The factor $\theta(s)$ comes from the fact that $\delta(X_s(t,u))=0$ for $s<0$. The fraction in (68) equals $+1$ except possibly at $s=0$. But since this point has zero measure, we replace this fraction with 1 and take the limit $\epsilon'\to 0$. The result is

$$F(0)\int_0^\infty ds\delta(s)=rF(0) \tag{69}$$

where $0<r<1$ is a number that results from integrating the δ-distribution on $\mathbb{R}^+$ rather then $\mathbb{R}$.

Case down

The calculation is almost identical to the *up* case, the only difference is that $\dot{e}^a(t_s)n_a^s(u_s)=-1$

for $s < 0$ and zero for $s > 0$. The result of the integral is

$$F(0)\int_{-\infty}^{0} ds\delta(s) = (1-r)F(0) \tag{70}$$

It is possible to fix the value of r by 1/2. The reason for is that if we change the orientations in a graph the up and down type edges interchange. If we consider the fact that the electric flux operator is one of the basic building block for defining geometric operators (e.g. volume or area) and we want these operators invariant under changing orientations, one must fix $r = 1 - r$.

We can summarize this calculation by defining $\epsilon(e, S)$ to be $+1, -1, 0$ when e is case *up,down* or *in(out)side* whence the value of (66) is

$$\frac{1}{2}\epsilon(e, S)F(0)$$

Using this result we are now able to calculate the Poisson bracket.

$$\begin{aligned}
\{E_n(S), h_e(A)\} = \frac{1}{2}\epsilon(e,S)\sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_0^1 dt_n \int_0^{t_n} dt_{n-1}\ldots\int_0^{k+2} dt_{k+1} \times \\
\times \int_0^0 dt_{k-1}\int_0^{k-1} dt_{k-2}\ldots\int_0^{t_2} dt_1 A(t_1)\ldots A(t_{k-1})\frac{n(b(e))}{2}A(t_{k+1})\ldots A(t_n) = \\
= \frac{1}{2}\epsilon(e,S)\frac{n(b(e))}{2}\left(1 + \sum_{n=2}^{\infty}\int_0^1 dt_n\int_0^{t_n} dt_{n-1}\ldots\int_0^{t_3} dt_2 A(t_2)\ldots A(t_n)\right) = \\
= \frac{1}{2}\epsilon(e,S)\frac{n(b(e))}{2}h_e(A)
\end{aligned} \tag{71}$$

where $n(x) = n^j(x)\tau_j$. In the first step we used the fact that the sum $\sum_{k=1}^{n}$ collapses to $k = 1$ because of the $\int_0^0 dt_{k-1}$ term and in the third step we have relabelled terms.

Formula (71) is our end result. Notice that the details of the regulation of the delta-distributions did not play any role. The important thing was that the regulation procedure included (piecewise) analytic and orientated curves and surfaces.

Now that we have derived the Poisson algebra of our fundamental variables it is time to define the basic operators of the theory.

Elementary operators

The most important property of the formula (71) is that it is again a product of finite number of holonomies, thus we can extend this expression to arbitrary cylindrical functions. Since in the process of calculating (71) we used smooth connections, we will first extend it to cylindrical functions restricted to smooth connections. Let $f \in Cyl(\overline{\mathcal{A}})$, then we find a subgroupoid $l = l(\gamma) \in \mathcal{L}$ and $f_l \in C^1(l)$ such that $f = p_l^* f_l = [f_l]_\sim$ and a complex valued function F_l on $G^{|E(\gamma)|}$ such that $f(A) = f_l(p_l(A)) = F_l(\rho_l(p_l(A)))$ where $\rho_l(A) = \{A(e)\}_{e\in E(\gamma)}$. We may chose γ in such a way that it is adapted to a given surface S, meaning all edges of γ are definite types with respect to S. If we restrict ourselves to $\mathcal{A}$ we obtain (using the chain rule)

$$\{E_n(S), f(A)\} = \frac{1}{2}\sum_{e\in E(\gamma)} \epsilon(e,S)\left[\frac{n(b(e))}{2}A(e)\right]_{AB}\frac{\partial F_l}{\partial A(e)}_{AB}(\{A(e')\}_{e'\in E(\gamma)}) \tag{72}$$

And because formula (72) leaves $C^\infty(X_l)$ restricted to $\mathcal{A}$ invariant, we can extend it to $\overline{\mathcal{A}}$. To simplify this let us introduce the *right- and left-invariant* vector fields on G.

$$\begin{aligned}
(R_j f)(h) &:= \left(\frac{d}{dt}\right)_{t=0} f(\exp(t\tau_j)h) =: \left(\frac{d}{dt}\right)_{t=0}[L^*_{\exp(t\tau_j)}f](h) \\
(L_j f)(h) &:= \left(\frac{d}{dt}\right)_{t=0} f(h\exp(t\tau_j)) =: \left(\frac{d}{dt}\right)_{t=0}[R^*_{\exp(t\tau_j)}f](h)
\end{aligned} \tag{73}$$

where $R_h(h') = h'h$ and $L_h(h') = hh'$ denotes the right and left action on G itself. The right (left) invariance of $R_j \quad (L_j)$, that is $(R_h)_* R_j = R_j \quad ((L_h)_* L_j = L_j)$ follows from from the commutativity of left and right translations $L_h R_{h'} = R_{h'} L_h$. Notice that the right-invariant field generates left translations and vice versa.

Using these fields we obtain

$$\{E^l_n(S), f_l(A)\} := \frac{1}{4}\sum_{e\in E(\gamma)} \epsilon(e,S) n_j R^j_e f_l \tag{74}$$

where R_e is R on the copy G labelled by e. So we may define a family of operators $\hat{E}^l_n(S)$ - *the electric flux operator* as

$$\hat{E}^l_n(S) f_l(A) := i\hbar\kappa\beta\{E^l_n(S), f_l(A)\} \tag{75}$$

Thus we defined a family of operators whenever $l(\gamma)$ is adapted to S. If not then we can produce an adopted one $l_S = l(\gamma')$, for example by choosing $r(\gamma') = r(\gamma)$ and by subdividing edges of γ into definite types with respect to S and the edges of γ' carry the orientation induced by the edges of γ. Then we simply define

$$p^*_{l_S l}(\hat{E}^l_n(S) f_l) := \hat{E}^{l_S}_n(S)(p^*_{l_S l} f_l) \tag{76}$$

One can check that this definition is consistent, that is (76) does not depend on the choice of adapted subgroupoid. Also one can verify the consistency check (*cylindrical consistency*)

$$p^*_{l'l}(\hat{E}^l_n(S) f_l) := \hat{E}^{l'}_n(S)(p^*_{l'l} f_l) \tag{77}$$

for all $l \prec l'$ which are not necessarily adapted. The importance of cylindrical consistency is that one can make calculations with finite degrees of freedom and it is not essential to know the projective limit. The consequence of (77) is that this family of operators defines a well-defined operator on $\mathcal{H}$.

The natural candidate for the coordinate operator would be the holonomy $h_e(A)$, but we need a little modification since this is a SU(2) valued quantity, thus $h_e(A)f(A)$ would not be a cylindrical function. Let $g : SU(2) \to \mathbb{C}$ be a complex valued function, then

$$\hat{g}(h_e(A))f(A) := g(h_e(A))f(A) \tag{78}$$

The function g depends on how we want to obtain the connection from the holonomy; it is usually the trace $(tr(h_e(A)))$.

What remains is to impose the reality conditions in the quantum regime. The statement that the connection and the electric flux are real is equivalent to the holonomy being unitary and the momentum operator self-adjoint.

3.2 Quantization of matter fields

We will see that the quantization of matter fields lie on the same line as in the case of the gravitational field. In fact the methods used there can be directly applied to vector fields and only minor modifications needed to quantize the scalar field. There are two reasons behind these appearing in a different section: 1) The differences in quantizing matter fields -

especially the U(1) case - will be crucial in the case of the Proca-field 2) It is worth separating this part because the notations might become confusing.

3.2.1 Vector field

Since the Ashtekar variables are SU(2) valued vector fields, one can simply adapt the results of section 3.1. The main difference will be that the commutative nature of the U(1) group will be strongly exploited. In this case our elementary quantities are the electric flux and holonomy, only this time the latter will be a U(1) valued object. Because of that one does not need the path ordered integration.

$$\begin{aligned} \underline{h}_e(\underline{A}) &:= \exp(i \int_e \underline{A}^c \dot{e}_c dt) \\ \underline{E}_S &:= \int_S (*\underline{E}) \end{aligned} \tag{79}$$

where $*\underline{E}_{ab} = \underline{E}^c \epsilon_{abc}$. The phase space consists of cylindrical functions defined as in 3.1 with the exception that these do not map from $SU(2)^{|E(\gamma)|}$, but from $U(1)^{|E(\gamma)|}$. The basis in this case is called *flux network state* and is much simpler than spin network states.

Definition 3.16 *Consider a γ graph and associate an integer $n_1, \ldots, n_{|E(\gamma)|}$ to each vertex. The flux network state is the following cylindrical function:*

$$F_{\gamma,\vec{n}} := \prod_{i=1}^{|E(\gamma)|} \underline{h}_{e_i}^{n_i}(\underline{A}) \tag{80}$$

Note that if the orientation of say e_i changes then the state remains the same if we replace n_i with $-n_i$. The corresponding operators are defined the same way as in the gravitational case

$$\begin{aligned} \hat{\underline{h}}_e F_{\gamma,\vec{n}} &:= \underline{h}_e F_{\gamma,\vec{n}} \\ \hat{\underline{E}}_S F_{\gamma,\vec{n}} &:= i\hbar\{\underline{E}_S, F_{\gamma,\vec{n}}\} \end{aligned} \tag{81}$$

which can be computed easily:

$$\hat{\underline{h}}_e F_{\gamma,\vec{n}} := F_{\gamma',\vec{n}'}$$

$$\underline{\hat{E}}_S F_{\gamma,\vec{n}} := -\hbar \sum_{i=1}^{|E(\gamma)|} n_i F_{\gamma,\vec{n}}$$

$$(\gamma', \vec{n}') = \begin{cases} (\gamma \cup e, \{n_1, \ldots, n_{|E(\gamma)|}, 1\}) & \text{if } \gamma \cap e = 0 \\ (\gamma, \{n_1, \ldots, n_i + 1, \ldots, n_{|E(\gamma)|}\}) & \text{if } e = e_i \in \gamma \end{cases} \tag{82}$$

The flux network states span the Hilbert-space of the electromagnetic field, which we will denote by $\mathcal{H}^{EM}$.

3.2.2 Scalar field

Quantizing the scalar field is difficult at first since one has the difficulty of finding a proper configuration variable, which is analogous to the holonomy. The problem is that there does not exist a covariant measure on the space of Φ-s. If there is a fixed background then there is no problem, one can define the usual Gaussian measure which leads to the Fock space representation. However a Gaussian measure for the scalar field is always background dependent [45]. The trick is to introduce the so-called *point holonomy* $U(\Phi(v)) := \exp(i\Phi(v))$ where v is a point in space. Now this is a U(1) valued object, thus - since U(1) is compact - we can use the Haar measure μ_H to construct a background independent measure. Formally this is of the form $d\mu_U := \prod_v d\mu_H(U(v))$ (of course this can be precisely defined on the same lines as the measure for the gravitational case). The quantum configuration space is simple: this is the space of generalized Higgs fields which are in bijection with all functions mapping from σ to U(1) - let us denote this space by $\mathcal{U}$. Then the Hilbert-space is $\mathcal{H}^{SK} := L_2(\mathcal{U}, d\mu_U)$. The basis is analogous to the flux network states - these are called *vertex functions*.

Definition 3.17 *Let γ be a graph and associate integers $m_1, \ldots, m_{|V(\gamma)|}$ to each of its vertices. Then the vertex function is defined as follows:*

$$D_{\gamma,\vec{m}} := \prod_{i=1}^{|V(\gamma)|} U(\Phi(v_i))^{m_i} \tag{83}$$

Of course this whole procedure can be extended to any compact group.
In the case of the U(1) case in [46] and [47] the authors argue that that the numbers m_i should be real instead of integers since in the second case one obtains periodic functions on the configuration space. However for real scalar fields these do not separate the points of the

configuration space. So to be more general we will use the following configuration operator and basis:

$$\begin{aligned} U(\Phi(v),\lambda) &:= \exp(i\lambda\Phi(v)) \\ D_{\gamma,\vec{\lambda}} &:= \prod_{i=1}^{|V(\gamma)|} U(\Phi(v_i))^{\lambda_i} \\ \lambda &\in \mathbb{R} \end{aligned} \tag{84}$$

What remains is to define the momentum operator. Since the configuration variable is not smeared, it is logical to start with the quantity

$$\Pi(B) := \int_B \Pi$$

where B is an open ball in σ (this is similar to the case of Ashtekar connection: the Poisson brackets are well defined on the Hilbert-space if the Poisson bracket of the corresponding variables are smeared in three dimensions). With this the momentum operator is defined as

$$\hat{\Pi}(B)D_{\gamma,\vec{\lambda}} := i\hbar\{\Pi(B), D_{\gamma,\vec{\lambda}}\} \tag{85}$$

which can be calculated in a similar fashion as the momentum operator of the electromagnetic field (due to the commutativity of the U(1) group).

3.3 Summary

To summarize if we consider a theory on curved space time with a U(1) scalar and also U(1) vector field, the Hilbert space will be of the form

$$\mathcal{H} := L_2(\mathcal{A}, d\mu_{AL}) \otimes L_2(\bar{\mathcal{A}}, d\mu_{EM}) \otimes L_2(\mathcal{U}, d\mu_U)$$

where $\mathcal{A}, \bar{\mathcal{A}}, \mathcal{U}$ are the generalized Ashtekar, electromagnetic connections and Higgs fields and $d\mu_{AL}, d\mu_{YM}, d\mu_U$ are the Ashtekar-Lewandowski, electromagnetic and U(1) measures respectively. On this Hilbert-space there exists a basis

$$B_{\gamma,\vec{I},\vec{\pi},\vec{n},\vec{\lambda}}(A, \underline{A}, \Phi) := T_{\gamma,\vec{I},\vec{\pi}}(A) \times F_{\gamma,\vec{n}}(\underline{A}) \times D_{\gamma,\vec{\lambda}}(\Phi)$$

which may be called the generalized spin network basis. This construction is independent of the Hamiltonian, it only depends on the gauge group of the fields. The next step is to rewrite the constraints in terms of the quantities which can be appointed to operators. This method is very similar to the one which we used to calculate the Poisson bracket between the holonomy and the electric flux.

4 Constraints

In this section we will first introduce the methods to rewrite the constraints to a form which consists of quantities to which we can associate an operator - this is called regularisation. The main challenge will be to do this in a background independent way, which means that if we e.g. do a point-splitting the way we do it must not be present in the final results. To define well-defined operators sometimes it will be necessary to rewrite the original constraints, for example in the case of the scalar constraint we need to eliminate the $\sqrt{q}$ from the denominator. Interestingly the result will contain the volume operator, which is why we will define and analyze this operator in a separate subsection. The details are important because all operators in Loop Quantum Gravity arise as a result of a regularisation but as we shall see in the case of the Proca-field or spontaneous symmetry breaking sometimes slight modifications of these results can obtain the final solution.
The second thing we wish to do is solve the constraints. As we will see sometimes the solutions are trivial (Gauss and diffeomorphism), but sometimes we can only manage to suggest a method which yields solutions (scalar constraint).

4.1 Gauge constraints

Implementation

First let us implement the gravitational Gauss constraint. The classical expression can be written in the form

$$G(\Lambda) := -\int d^3x (D_a\Lambda^J)E^a_J = -E(D\Lambda) \tag{86}$$

where $D_a\Lambda^J := \partial_a\Lambda^J + f^J_{KL}A^K_a\Lambda^L$ is the covariant derivative of the smearing field Λ. Notice that (86) is an electric field smeared in three dimensions except that the smearing field depends on the configuration space. Nevertheless the results obtained in section 3.1 can be applied here as well since the during the regularisation of the Poisson bracket we considered general smearing fields. After that we will extend the result to cylindrical functions via the chain rule and hope that the final result is also a cylindrical function.
We will not write down all the steps since it is completely the same as in section 3.1, the only

difference is that the $\lim \epsilon' \to 0$ is missing. Further more the splitting of edges to different types is not necessary because E is smeared in three dimensions. The result is

$$\{E(D\Lambda), h_p(A)\} = \beta\kappa \int_0^1 dt\dot{p}^a(t)[D_a\Lambda^J](p(t))h_{p[0,t]}(A)\frac{\tau_J}{2}h_{p[t,1]}(A). \tag{87}$$

Let us use the notation $A(p(t)) := \dot{p}^a(t)A_a^J(p(t))\frac{\tau_J}{2}$ and $[\tau_J, \tau_K] = 2f_{JK}^L\tau_L$ to cast this expression in the following form:

$$\{E(D\Lambda), h_p(A)\} = \frac{\beta\kappa}{2}\int_0^1 dth_{p[0,t]}(A)\left\{\frac{d}{dt}\Lambda(p(t)) + [A(p(t)), \Lambda(p(t))]\right\}h_{p[t,1]}(A) \tag{88}$$

Now if we use the parallel transport equation for the holonomy

$$\frac{d}{dt}h_{p[0,t]}(A) = h_{p[0,t]}(A)A(p(t))$$

and the identity $h_{p[t,1]}(A) = h_{p[0,t]}(A)^{-1}h_p(A)$ we can see that (88) becomes

$$\begin{aligned}\{E(D\Lambda), h_p(A)\} &= \frac{\beta\kappa}{2}\int_0^1 dt\frac{d}{dt}[h_{p[0,t]}(A)\Lambda(p(t))h_{p[t,1]}(A)] = \\ &= \frac{\beta\kappa}{4}[-\Lambda(b(p))h_p(A) + h_p(A)\Lambda(f(p))] \end{aligned} \tag{89}$$

where we have performed an integration by parts in the last step. We can see that luckily this is a cylindrical function, so we can extend this expression to any element of the Hilbert space. Let γ be a graph and for any subgroupoid $l = l(\gamma)$ we obtain

$$\begin{aligned}\{G(\Lambda), f_l\} &= -\frac{\beta\kappa}{4}\sum_{e\in E(\gamma)}[\Lambda(b(e))A(e) - A(e)\Lambda(f(e))]_{AB}\frac{\partial f_l}{\partial A(e)_{AB}}(A) = \\ &= -\frac{\beta\kappa}{4}\sum_{e\in E(\gamma)}\left[\Lambda_J(b(e))R_e^J - \Lambda_J(f(e))L_e^J\right]f_l(A) \end{aligned} \tag{90}$$

We can write this expression as a sum over vertices in a compact form

$$\{G(\Lambda), f_l\} := -\frac{\beta\kappa}{4}\sum_{v\in V(\gamma)}\Lambda_J(v)\left[\sum_{e\in E(\gamma);v=b(e)}R_e^J - \sum_{e\in E(\gamma);v=f(e)}L_e^J\right]f_l(A). \tag{91}$$

Since Λ is real-valued for SU(2), this expression is also real-valued. Also using the same analysis as for the momentum operator one finds that it is a consistent family and it is μ-compatible. Because of this the corresponding operator defined as

$$\hat{G}_l(\Lambda)f_l := \frac{-i\beta l_P^2}{4}\sum_{v\in V(\gamma)}\Lambda_J(v)\left[\sum_{e\in E(\gamma);v=b(e)} R_e^J - \sum_{e\in E(\gamma);v=f(e)} L_e^J\right] f_l(A). \tag{92}$$

is essentially self-adjoint.

To see that this operator can be interpreted as the quantum version of gauge transformation, let us consider an infinitesimal gauge transformation $g_t(x) := \exp(t\Lambda_J(x)\tau_j)$ for some functions $\Lambda_J(x)$ and $t \to 0$. For simplicity let us consider the case where $\Lambda_J(x)$ is non-zero only in vertex v. Now consider a spin network function of the form

$$T_s := \left[\prod_{e\in\gamma(s),f(e)=v} f_e(h_e)\right]\left[\prod_{e\in\gamma(s),b(e)=v} f_e(h_e)\right] F_s$$

where F_s is a cylindrical function that does not depend on the edges incident at v. Then under an infinitesimal gauge transformation this spin network function transforms as

$$\begin{aligned}
&\left(\frac{d}{dt}\right)_{t=0}\lambda_{g_t}^* T_s = \\
&= \left(\frac{d}{dt}\right)_{t=0}\left[\prod_{e\in\gamma(s),f(e)=v} f_e(g_t(v)h_e)\right]\left[\prod_{e\in\gamma(s),b(e)=v} f_e(h_e g_t^{-1}(v))\right] F_s = \\
&= \left(\frac{d}{dt}\right)_{t=0}[\circ_{e\in\gamma(s),f(e)=v}(L^e_{g_t(v)})^*] \circ [\circ_{e\in\gamma(s),b(e)=v} R^e_{g_t^{-1}(v)})^*] T_s = \\
&= \Lambda_J(v)\left[\sum_{e\in E(\gamma),f(e)=v} R_e^J - \sum_{e\in E(\gamma),b(e)=v} L_e^J\right] T_s = \\
&= G_{l(\gamma(s))}(\tilde{\Lambda})[T_s]
\end{aligned} \tag{93}$$

A completely same analysis can be done to derive the operator corresponding to the electromagnetic Gauss constraint. Here

$$\underline{G}(\Lambda) := -\int d^3x(\partial_a\Lambda)\underline{E}^a = -\underline{E}(D\Lambda) \tag{94}$$

The Poisson bracket is simpler since we can exploit the commutative nature of the U(1) group. If $f_l \in \mathcal{H}^{EM}$ is a cylindrical function then we have

$$\underline{\hat{G}}_l(\Lambda) f_l(\underline{h}(\underline{A})) := -i \sum_{v \in V(\gamma)} \Lambda(v) \frac{\partial}{\partial \underline{h}_e} f_l(\underline{h}(\underline{A})). \tag{95}$$

Applying this operator to a flux network state we obtain the simple result

$$\underline{\hat{G}}_l(\Lambda) F_{\gamma,\vec{n}}(\underline{h}(\underline{A})) := \sum_{v \in V(\gamma)} \Lambda(v) \sum_{e \in E(\gamma); e \cap v = v} n_e F_{\gamma,\vec{n}}(\underline{h}(\underline{A})). \tag{96}$$

Solution

First let us derive the algebra of $\hat{G}(\Lambda)$. Using the Lie-algebra of the left- and right-invariant vector fields given by

$$[R_e^J, R_{e'}^K] = -2\delta_{ee'} f_L^{JK} R_e^L, \ [L_e^J, L_{e'}^K] = -2\delta_{ee'} f_L^{JK} L_e^L, \ [R_e^J, L_{e'}^K] = 0$$

we find

$$\begin{aligned} &[\hat{G}_l(\Lambda), \hat{G}_l(\Lambda')] = \\ &= \left(\frac{\beta\kappa}{4}\right)^2 \sum_{e \in E(\gamma)} \{\Lambda_J(b(e))\Lambda'_K(b(e))[R_e^J, R_e^K] + \Lambda_J(f(e))\Lambda'_K(f(e))[L_e^J, L_e^K]\} = \\ &= -\frac{\beta\kappa}{2} G_l([\tilde{\Lambda}, \tilde{\Lambda}']), \end{aligned} \tag{97}$$

where we introduced the notation $\tilde{\Lambda}(x) := \Lambda_J(x)\tau_J/2$. Now according to the RAQ program we are looking for distributions L such that for every cylindrical function f_l

$$L(p_l^* \hat{G}_l(\tilde{\Lambda}) f_l) = 0 \tag{98}$$

Since Λ_J is arbitrary we may restrict it to one vertex only, thus the condition (98) is completely equivalent to

$$L(p_l^* \left[\sum_{e \in E(\gamma); v = b(e)} R_e^J - \sum_{e \in E(\gamma); v = f(e)} L_e^J \right] f_l) = 0 \tag{99}$$

To continue we use the fact that each L can be written as

$$L = \sum_{s \in S} < T_s, \ldots >,$$

where $<, >$ is the inner product on the Hilbert space, T_s are spin network functions and S denotes the set of all spin network labels ($s = \{\gamma, \vec{\pi}, \vec{I}\}$). Since the spin network functions form a complete orthonormal basis (99) is equivalent to

$$L(p^*_{l(\gamma(s))} \left[\sum_{e \in E(\gamma(s)); v=b(e)} R^J_e - \sum_{e \in E(\gamma(s)); v=f(e)} L^J_e \right] T_s) = 0 \tag{100}$$

for any $v \in V(\gamma(s))$ where $\gamma(s)$ is the graph that underlies s. Since the operator involved in (99) leaves $\vec{\pi}(s)$ and $\gamma(s)$ invariant we have

$$\sum_{s' \in S, \gamma(s') = \gamma(s), \vec{\pi}(s') = \vec{\pi}(s)} < T_{s'}, \left[\sum_{e \in E(\gamma(s)); v=b(e)} R^J_e - \sum_{e \in E(\gamma(s)); v=f(e)} L^J_e \right] T_s >= 0 \tag{101}$$

Effectively the sum over s' is now reduced over all intertwiners while $\gamma', \vec{\pi}(s')$ are fixed, thus we have a *finite* sum over the intertwiners. In theory we could proceed by solving this system of linear equations but there is a simpler method. If we recall that the action of the operator in (101) is the infinitesimal generator of the Gauss constraint then it is clear that the solution to the constraint is of the form where all $T_{s'}$ are gauge invariant spin network functions. One can show that a spin network function is gauge invariant if and only the intertwiner projects to the trivial representation. The gauge invariant spin network functions span a subspace of the Hilbert space.

The U(1) case is much simpler due to the commutativity of the group. The steps can be repeated in the same way as in the SU(2) case, arriving to the equation

$$\sum_{s' \in S, \gamma(s') = \gamma(s), \vec{n}(s') = \vec{n}(s)} < F_{s'}, \sum_{e \in E(\gamma); e \cap v = v} n_e F_s >= 0 \tag{102}$$

where $\overline{T}_{s'}$ are the coefficients of the solution functional and F_s are the flux network states. The solution of this equation is simple: the gauge invariant flux network states will be those where for each vertex v the sum of integers for the incoming edges equals the sum of integers for the outgoing edges.

4.2 Diffeomorphism constraint

Implementation

Spatial diffeomorphisms have a natural implementation on cylindrical functions, one only has to lift the action on holonomies

$$U_{diff}(\phi)f_\gamma(A(e)) := f_\gamma(A(\phi^{-1}(e))) \tag{103}$$

which for spin network functions means that one maps the graph $\gamma(s)$ to $\phi^{-1}(s)$ while the labels $\vec{\pi}, \vec{I}$ are carried from e and v to $\phi^{-1}(e)$ and $\phi^{-1}(v)$ respectively. But we want to implement the constraint at the quantum level the same way as we did for the Gauss constraint. Consider the classical diffeomorphism constraint (modulo gauge transformations)

$$V_a = H_a - A_a^J G_J = 2(\partial_{[a}A_{b]})E_J^b - A_a^J \partial_b E_J^b \tag{104}$$

and smear it with a vector field u:

$$V(u) := \int d^3x (\mathcal{L}_u A^J)_a(x) E_J^a(x) = E(\mathcal{L}_u A). \tag{105}$$

The calculation is similar as in the case of the Gauss constraint. The Poisson bracket with a holonomy yields

$$\{V(u), h_p(A)\} := \beta\kappa \int_0^1 dt h_{p[0,t]}(\mathcal{L}_u A)(p(t)) h_{p[t,1]}. \tag{106}$$

Now consider the semi-analytic diffeomorphisms ϕ_t^u which are determined by the integral curves of u, that is, are solutions to the differential equation

$$\dot{c}_{u,x}(t) = u(c(t)) \qquad c_{u,x}(0) = x$$

with $\phi_t^u(x) := c_{u,x}(t)$. We claim that the quantity (106) equals

$$\left(\frac{d}{dt}\right)_{t=0} h_p((\phi_t^u)^* A).$$

To see this one uses the expansion $(\phi_t^u)^* A = A + (\mathcal{L}_u A)t + O(t^2)$ and the fact that if $p = p_1 \circ \ldots \circ p_N$ then $h_p = h_{p_1} h_{p_2} \ldots h_{p_N}$ with $p_k := p([t_{k-1}, t_k])$, $0 = t_0 < t_1 < \ldots < t_N = 1, \quad t_k - t_{k-1} < 1/N$. If we denote $\delta h_{p_k}(A) = h_{p_k}(A + \delta A) - h_{p_k}(A)$ then

$$\begin{aligned} & h_p(A+\delta A) - h_p(A) = \\ = & \sum_{n=1}^{N} \sum_{1 \leq k_1 \leq \ldots \leq k_n \leq N} (h_{p_1 \circ \ldots p_{k_1}-1}(A) \delta h_{p_{k_1}})(h_{p_{k_1+1} \circ \ldots p_{k_2}-1}(A) \delta h_{p_{k_2}}) \times \ldots \\ \ldots & \times (h_{p_{k_{n-1}+1} \circ \ldots p_{k_n}-1}(A) \delta h_{p_{k_n}})(h_{p_{k_n+1} \circ \ldots p_N}(A)) \end{aligned} \tag{107}$$

which holds at each finite N. Now using the formula $h_{p_k}(A) = \mathcal{P}\exp(\int_{p_k} A^J \tau_J/2)$ we obtain

$$\delta h_{p_k} = \mathcal{P}\{e^{[A+\delta A]_{p_k}} - e^{A_{p_k}}\} \tag{108}$$

where $A_{p_k} = \int_{p_k} A^J \tau_J/2$. This means that δh_{p_k} is at least linear in δA and therefor in t. Thus if we divide (107) with t and take the limit $t \to 0$ we get

$$\left(\frac{d}{dt}\right)_{t=0} h_p((\phi_t^u)^* A) = \sum_{n=1}^{N} h_{p_1 \circ \ldots p_{n-1}}(A) \left(\frac{d}{dt}\right)_{t=0} h_{p_n}((\phi_t^u)^* A) h_{p_{n+1} \circ \ldots p_N}(A). \tag{109}$$

Finally if we consider that $h_{p_k}(A+\delta A) - h_{p_k}(A) = \delta A_{p_k} + O(1/N^2)$, in the limit $t \to 0$ (109) turns into (106).

Unfortunately (106) is no longer a cylindrical function and therefore we cannot construct a consistent family of cylindrically defined vector fields for the diffeomorphism constraint. In other words, (106) cannot be extended to $\overline{\mathcal{A}}$. Of course for each t the function $h_{p[0,t]}$ in (106) can be extended to $\overline{\mathcal{A}}$, but $\mathcal{L}_u(A)$ only makes sense for smooth A. Thus we are not able to define an operator that corresponds to the infinitesimal diffeomorphism constraint. This does not change if one adds matter fields to the constraint, so we need an alternative solution.

The way out is the observation that *finite diffeomorphisms* can be extended to $\overline{\mathcal{A}}$. In fact the identity $\{V(u), h_p(A)\} = \left(\frac{d}{dt}\right)_{t=0} h_p((\phi_t^u)^* A)$ suggests considering the exponentiated quantity $\exp(\{V(u), A\})$ which then gives the action $h_p(A) \to h_p((\phi_t^u)^* A)$. Since classically we can always recover the infinitesimal action from the exponentiated one, we do not lose any information. Moreover, we may consider general finite diffeomorphisms ϕ which unlike the ϕ_t^u are not necessarily connected to the identity. Now we have for smooth A

$$h_p(\phi^* A) = \mathcal{P}\exp(\int_p \phi^* A) = \mathcal{P}\exp(\int_{\phi(p)} A) = h_{\phi(p)}(A) \tag{110}$$

which is the action of diffeomorphisms to paths.

Solution

We have seen that it is impossible to define a unitary representation of the diffeomorphism group on the Hilbert space. This behavior is drastically different that of Gaussian measures and is deeply rooted to the background independence of our theory: the covariance of the measure depends on a background metric which enables us to tell how far points are. However in a diffeomorphism invariant theory there is no distinguished background metric but only diffeomorphisms which can take two points as far apart or close together as we desire. But we have also shown that one may use finite diffeomorphisms instead to solve the constraints. Before we continue with the solution it has to be stressed out that - though the solution, as we shall see, is self-consistent - there still remain a few questions about treating the diffeomorphism constraint in such a fashion. The main problem is that qualitatively one may think that near the Planck-scale diffeomorphism symmetry is somehow broken since it is meaningless to search for the quantum version of this constraint. It is likely that at the Planck-scale there is a purely combinatorial "symmetry", which - at large scales - becomes diffeomorphism invariance.
Now let $\hat{U}_{diff}(\phi)$ be the generator for the diffeomorphism ϕ. We are searching for algebraic distributions L such that for every spin network function

$$L(\hat{U}_{diff}(\phi)T_s) = 0. \tag{111}$$

If we use the fact that every L can be written in the form $L = \sum_s L_s < T_s, . >$ then (111) becomes a very simple condition on the coefficients L_s:

$$L_{\phi \cdot s} = L_s. \tag{112}$$

This suggests introducing the orbit $[s]$ of s defined by

$$[s] = \{\phi \cdot s; \phi \in Diff(\sigma)\} \tag{113}$$

thus condition (112) means that L_s is constant on every orbit. Obviously $\mathcal{S}$ (the set of spin networks) is a the disjoint union of orbits which motivates us to introduce the space of orbits $\mathcal{N}$ whose elements we denote by ν. Introducing the elementary distribution $L_\nu = \sum_{s\in\nu} < T_s, . >$ we may write the general solution of the diffeomorphism constraint as

$$L = \sum_{\nu\in\mathcal{N}} c_\nu L_\nu \tag{114}$$

for some complex coefficients c_ν which depend only on the orbit not the representation. Notice also that $L_\nu(T_s) = \chi_\nu(s)$ where χ denotes the characteristic function.
It is straightforward to generalize this if matter fields are present, one only needs to replace s with $\tilde{s}$, where

$$\tilde{s} := \{\gamma, \vec{\pi}, \vec{I}, \vec{\lambda}\}$$

is the *generalized spin network*.

4.3 Scalar constraint

4.3.1 Regularisation of the scalar constraint

Implementing and solving the Hamiltonian constraint is the most important task since this is where the true physics is encoded. Unfortunately this constraint is the most difficult one to solve which is because 1. it is extremely non-linear and 2. the Dirac algebra is not a Lie algebra due to the structure functions. The complex Ashtekar variables were originally introduced because using these the rescaled Hamiltonian $\sqrt{q}H$ is then only polynomial. Unfortunately in this case we arrive to a non-compact group for the holonomies thus the Hilbert space defined earlier cannot be used. Also it turns out that it is impossible to construct a background-independent operator-valued distribution corresponding to $\sqrt{(}q)H$. The reason is that this quantity is a density of weight two while - as we shall see later - *only density of weight one have a chance to result in a well-defined operator.*
The way out of this is to rewrite the constraint which is more suitable to our cause. Our

elementary variables are A and E, thus one avoids using $K_a^I = A_a^I - \Gamma_a^I$, however we will first rewrite the scalar constraint in the following way [32]:

$$H = \frac{4}{\kappa\sqrt(q)} tr([K_a, K_b][E^a, E^b]) - H_E, \tag{115}$$

where

$$H_E = \frac{2}{\kappa\sqrt(q)} tr(F_{ab}[E^a, E^b]) \tag{116}$$

is called the *Euclidean Hamiltonian constraint* because it would be the Hamiltonian constraint for canonical Euclidean gravity. Note that we have introduced the factor $1/\kappa$ which will get the dimensionalities right and we have used the notations $F_{ab} = F_{ab}^J \tau_j/2,\ E^a = E_J^a \tau_J/2,\ K^a = K_J^a \tau_J/2,\ A^a = A_J^a \tau_J/2$. Now consider the following two quantities:

1. The volume of the region R in σ is

$$V(R) := \int_R d^3x \sqrt{(|det(q)|)}$$

2. The integrated densitised trace of the extrinsic curvature

$$K := \int_\sigma d^3x K_a^I E_I^a.$$

Notice that in the case of the volume we have used the absolute vale of the determinant. At the classical level this is not necessary because $det(q)$ is always positive, however at the quantum level we must allow the changing of $sgn(det(q))$ otherwise E cannot become a derivative operator. This will not be a problem if we consider physical states since these will be peaked on constant sign. Now the following two classical identities will be the key why we have written the constraint in a different way:

$$\begin{aligned} sgn(det(e))\frac{E_K^a E_L^b \epsilon_{JKL}}{\sqrt{det(q)}} &= \epsilon^{abc} e_c^J = 2\epsilon^{abc}\frac{\delta V(R)}{\delta E_J^a} = \frac{4}{\kappa}\{V(R), A_a^J\} \\ K_a^J &= \frac{\delta K}{\delta E_J^a} = \frac{2}{\kappa}\{K, A_a^J\}. \end{aligned}$$

Using these identities H and H_E can be written in the following way:

$$sgn(det(e))(H - H_E) = -\frac{128}{\kappa}\epsilon^{abc}tr(\{A_a, K\}\{A_b, K\}\{A_c, V(R)\}) \quad (117)$$

$$sgn(det(e))H_E = -\frac{8}{\kappa}\epsilon^{abc}tr(F_{ab}\{A_c, V(R)\}) \quad (118)$$

or in integrated form (N is some lapse function)

$$(H - H_E)(N) = -\frac{128}{\kappa}\int_\sigma N' tr(\{A, K\} \wedge \{A, K\} \wedge \{A, V(R)\}) \quad (119)$$

$$H_E(N) = -\frac{8}{\kappa}\int_\sigma N' tr(F \wedge \{A, V(R)\}) \quad (120)$$

Here we absorbed the $sgn(det(e))$ into N and denoted it N'. In what follows we will drop the prime. What we have achieved is that we removed the problematic $1/\sqrt{det(q)}$ from the denominator by means of Poisson brackets. The advantage of this is if we quantize $\sqrt{det(q)}$ - which is the volume operator (see Appendix A) - we find it has a non-trivial kernel [50], thus the quantization of the original constraint would be problematic. But now we do not have this problem, so we may proceed with the quantization.

First we have to express (119) and (120) in terms of $A(e)$. We can do this by introducing a triangulation $T(\epsilon)$ of σ by tetrahedra which fill all of σ and intersect each other only in lower dimensional submanifolds of σ. The parameter ϵ indicates how fine is the triangulation, the limit $\epsilon \to 0$ corresponds to tetrahedra with vanishing volume (the number of tetrahedra grows in this limit so as to always fill out σ). So let $e_I(\Delta)$ denote the edges of the tetrahedron $\Delta \in T(\epsilon)$ and let $v(\Delta)$ be the common intersection point. The matrix consisting of the tangents of $e_1(\Delta), e_2(\Delta), e_3(\Delta)$ at $v(\Delta)$ (in that sequence) has non-negative determinant which induces an orientation of Δ. Further more let a_{IJ} be the arc on the boundary of Δ connecting the endpoints of $e_I(\Delta), e_J(\Delta)$ such that the loop $\alpha_{IJ}(\Delta) = e_I(\Delta) \circ a_{IJ}(\Delta) \circ e_J(\Delta)^{-1}$ has positive orientation for $(I, J) = (1, 2), (2, 3), (3, 1)$ and negative otherwise. One then can see that in the limit as $\epsilon \to 0$ the quantities

$$(H^\epsilon - H_E^\epsilon)(N) = \frac{128}{3\kappa^4}\sum_{\Delta \in T(\epsilon)} \epsilon^{IJK} N(v(\Delta)) \times$$
$$\times\ tr(h_{e_I(\Delta)}\{h^{-1}_{e_I(\Delta)}, K\}h_{e_J(\Delta)}\{h^{-1}_{e_J(\Delta)}, K\}h_{e_K(\Delta)}\{h^{-1}_{e_K(\Delta)}, V(R_{v(\Delta)})\}) \quad (121)$$

$$H_E^\epsilon(N) = \frac{8}{3\kappa^2}\sum_{\Delta \in T(\epsilon)} \epsilon^{IJK} N(v(\Delta)) tr(h_{\alpha_{IJ}(\Delta)} h_{e_K(\Delta)}\{h^{-1}_{e_K(\Delta)}, V(R_{v(\Delta)})\}) \quad (122)$$

converge to (119) and (120) respectively point-wise on $\mathcal{M}$ for any choice of triangulation. This independence of the limit from the choice of triangulation enables us to chose it state-dependent, that is we may adept the triangulation to the graph of the state.

In order to verify the limits one makes use of the following facts. First let e, e' be arbitrary

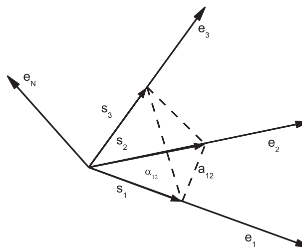

Figure 5: The meaning of a tetrahedron, segments and arcs at a vertex.

paths which are images of the interval $[0,1]$ under the correspondent embeddings, which we also denote by e, e' and $e(0) = e'(0) = v$. For any $0 < \epsilon < 1$ set $e_\epsilon(t) := e(\epsilon t)$ for $t \in [0,1]$ and likewise for e'. Then we expand $h_{e_\epsilon}(A)$ in powers of ϵ. It is not difficult to see that $h_{e_\epsilon}(A) = 1 + \epsilon \dot{e}(0) A_a^J(v)\tau_J/2 + O(\epsilon^2)$. Next consider the loop $\alpha_{e_\epsilon, e'_\epsilon}$ where

$$\alpha_{e_\epsilon, e'_\epsilon} = \begin{cases} e_\epsilon(4t) & 0 \le t \le 1/4 \\ e_\epsilon(1) + e'_\epsilon(4t-1) - v & 1/4 \le t \le 1/2 \\ e'_\epsilon(1) + e_\epsilon(3-4t) - v & 1/2 \le t \le 3/4 \\ e'_\epsilon(4-4t) & 3/4 \le t \le 1 \end{cases}$$

Now expanding again in powers of ϵ we obtain that $h_{\alpha_{e_\epsilon, e'_\epsilon}} = 1 + \epsilon^2 F_{ab}^J \dot{e}^a(0) \dot{e'}^b(0) \tau_J/2 + O(\epsilon^3)$. Since constants drop out of Poisson brackets we see that the Poisson bracket in (122) is of order ϵ while the loop contribution is of order ϵ^2 giving a total ϵ^3 at lowest order, which is

precisely the order we need to recast (122) into a Riemann sum approximation of the continuum integral.
Now if the operators $\hat{V}$ and $\hat{K}$ exist we can simply replace the Poisson brackets by commutators times $1/(i\hbar)$ and arrive to well-defined operators. In Appendix A the reader will see that one can define the volume operator, which is crucial not only for the gravitational, but for the Maxwell and scalar Hamiltonians as well.
What remains is the definition of $\hat{K}$. Recall the classical identity that the integrated densitised trace of the extrinsic curvature is the 'time derivative' of the total volume, i.e.

$$K = -\{H_E(1), V(\sigma)\}$$

where N=1 is the constant lapse. Since the operators $\hat{H}_E(N)$ and $\hat{V}$ exist then one can define

$$\hat{K} := \frac{i}{\hbar}[\hat{H}_E(1), \hat{V}],$$

which means that we can define $\hat{H}$ in a similar fashion.

4.3.2 Properties of the scalar constraint operator

Before we proceed to solving the constraint we need to exam the properties of the operator defined in the previous section. This is important since we have to know whether these operators are well-defined and can they be implemented as quantum versions of the classical constraint. There are three main questions that should be answered:
I. What are the allowed , physically relevant choices for the family of triangulations $T(\epsilon)$?
II. How should we treat the limit $\epsilon \longrightarrow 0$ for $\hat{H}^\epsilon$?
III. What is the commutator algebra of these operators? Is it anomaly free?

To answer the first question we define the natural choice of triangulation: given a graph γ we construct a triangulation $T(\gamma, \epsilon)$ of σ *adapted* to γ which satisfies the following requirements:

1. The graph γ is embedded in $T(\gamma, \epsilon)$ for all $\epsilon > 0$.

2. The valance of each vertex of γ remains constant for all $\epsilon > 0$.

3. Choose a system of semi-analytic arcs $a^{\epsilon}_{\gamma,v,e,e'}$, one for each pair of edges e, e' of γ incident at a v vertex of γ, which do not intersect γ except in its endpoints where they intersect transversally. These endpoints are interior points of e, e' and between $e \cap a^{\epsilon}_{\gamma,v,e,e'}$, $e' \cap a^{\epsilon}_{\gamma,v,e,e'}$ and v there are no vertices of γ. For each ϵ, ϵ' the arcs $a^{\epsilon}_{\gamma,v,e,e'}$ and $a^{\epsilon'}_{\gamma,v,e,e'}$ are diffeomorphic. The segments of e, e' incident at v with outgoing orientation that are determined by the endpoints of the arc $a^{\epsilon}_{\gamma,v,e,e'}$ are denoted by $s^{\epsilon}_{\gamma,v,e}$ and $s^{\epsilon}_{\gamma,v,e'}$ respectively.

4. Choose a system of mutually disjoint neighborhoods $U^{\epsilon}_{\gamma,v}$, one for each vertex v, and require that for each $\epsilon > 0$ the arcs $a^{\epsilon}_{\gamma,v,e,e'}$ are contained in $U^{\epsilon}_{\gamma,v}$. These neighborhoods are nested in a sense that $U^{\epsilon}_{\gamma,v} \subset U^{\epsilon'}_{\gamma,v}$ if $\epsilon < \epsilon'$ and $\lim_{\epsilon \to 0} U^{\epsilon}_{\gamma,v} = \{v\}$.

5. Triangulate $U^{\epsilon}_{\gamma,v}$ by tetrahedra $\Delta^{\epsilon}_{\gamma,v,e,e',\tilde{e}}$, one for each ordered triple of distinct edges $e, e', \tilde{e}$ incident at v, bounded by the segments $s^{\epsilon}_{\gamma,v,e} s^{\epsilon}_{\gamma,v,e'}$, $s^{\epsilon}_{\gamma,v,\tilde{e}}$ and the arcs $a^{\epsilon}_{\gamma,v,e,e'}$, $a^{\epsilon}_{\gamma,v,e',\tilde{e}}$, $a^{\epsilon}_{\gamma,v,\tilde{e},e}$ from which loops $\alpha^{\epsilon}_{\gamma,v,e,e'}$, etc. are built. The ordered triple $e, e', \tilde{e}$ is such that their tangents at v, in this order, form a matrix of positive determinant.

Requirement (1) prevents the action of the Hamiltonian constraint operator from being trivial. Requirement (2) guarantees that the regulated operator $\hat{H}^{\epsilon}(N)$ is densely defined for each ϵ. The remaining requirements specify the triangulation for each vertex of γ and leave it unspecified outside of them. One can show that triangulations satisfying these requirements always exist, further more in [32] it was shown how to deal with degenerate situations (for example how to construct a tetrahedron for planar vertices). The reason why those tetrahedra that lie outside the neighborhoods of the vertices described above are irrelevant rests crucially of the choice of ordering the Hamiltonian with $[\hat{h}_s^{-1}, \hat{V}]$ on the rightmost and our choice for the volume operator: an important property of the volume operator derived in Appendix A is that it annihilates states that have vertices consisting of planar edges. Let us consider a cylindrical function f over a graph γ and let s be such that it has support outside of each vertex of γ. In this setup the set $V(\gamma \cup s) - V(\gamma)$ consists of planar and at most four-valent vertices which means that $[\hat{h}_s^{-1}, \hat{V}]f = 0$. However if we consider the volume operator derived in [20] which does not annihilate planar vertices $[\hat{h}_s^{-1}, \hat{V}]f$ would not be zero even for trivalent vertices. In other words in the limit of small ϵ the operator would maps us out of the space of cylindrical functions. In summary one could say that dynamics 'happens only at the vertices of the graph'.

What we obtained is a family of operators $\hat{H}^{\epsilon}_{\gamma}(N)$ since we have adapted the regularisation to graph of the state on which the operator acts. One may worry that this does not define a cylindrically consistent operator but fortunately this is not the case since this operator is well-defined on spin network functions and its action on this basis is a linear combination of spin network functions.
The operator we defined is by construction background dependent but not symmetric, which is not necessary for a constraint operator. In fact some argue ([22],[23]) that the constraint should not be symmetric in order the constraint algebra be non-anomalous.
Finally we should point out that there are still a lot of ambiguities in the regularisation process. For instance the factor 1/3 appearing in the constraints come from the fact that we used tetrahedron for elementary cells. But this is not necessary, one also could use cubic etc. regularisation schemes. Also instead of using $tr(\tau_J h_s)$ one could use

$$\frac{\sum_{k=1}^{N} tr(\tau_J h_s^{n_k})}{\sum_{k=1}^{N} n_k}$$

since in leading order we get the same result. Whether there is a natural (physical) choice for a particular regularisation process is yet unknown.
Remark: There are a lot of issues which one has to consider to obtain a well defined quantization, but the detailed analysis would cover much space and these have little connection to the results of the massive vector fields. Here we just mention some of these:
1) *Operator limit:* Taking the limit $\epsilon \to 0$ turns out to be a bit technical but has a simple solution: one simply drops the parameter ϵ from the expressions.
2) *Dirac algebra:* The question is whether the commutator between two Hamiltonian constraints and between Hamiltonian and diffeomorphism constraints exists and is free of anomalies. Using a few tricks the second can be proven quite easily but the first is a bit problematic: the proof very much depends on the choice of regularisation *and* the volume operator, and the result is not satisfactory either, so we can say that this issue is not closed yet.

4.4 Solution of the scalar constraint

Before we solve the constraint let us examine how it acts on a spin network function $T_{\gamma,\vec{\pi},\vec{I}}$. It is given by

$$
\begin{aligned}
(\hat{H}^\epsilon - \hat{H}^\epsilon_E)(N)T_{\gamma,\vec{\pi},\vec{I}} &= \frac{128}{3\kappa(il_P^2)^3}\sum_{v\in V(\gamma)}\frac{N(v)}{E(v)}\sum_{v(\Delta)=v}\epsilon^{IJK}tr(h_{e_I(\Delta)}[h^{-1}_{e_I(\Delta)},\hat{K}^\epsilon]\times \\
&\times\ h_{e_J(\Delta)}[h^{-1}_{e_J(\Delta)},\hat{K}^\epsilon]h_{e_K(\Delta)}[h^{-1}_{e_K(\Delta)},\hat{V}(U^\epsilon(v))])T_{\gamma,\vec{\pi},\vec{I}} \qquad (123) \\
\hat{H}^\epsilon_E(N)T_{\gamma,\vec{\pi},\vec{I}} &= \frac{8}{3i\kappa l_P^2}\sum_{v\in V(\gamma)}\frac{N(v)}{E(v)}\sum_{v(\Delta)=v}\epsilon^{IJK}\times \\
&\times\ tr(h_{\alpha_{IJ}(\Delta)}h_{e_K(\Delta)}[h^{-1}_{e_K(\Delta)},\hat{V}(U^\epsilon(v))])T_{\gamma,\vec{\pi},\vec{I}} \qquad (124)
\end{aligned}
$$

Here the factor $E(v)=n(v)(n(v)-1)(n(v)-2)/6$ is a combinatorial factor coming from the fact that we adopted the triangulation to the graph. First let us look at the Euclidean part. When it acts on a spin network state it looks at each non-planar vertex of the graph γ and considers each triple of edges incident at it (e,e',e''). For each such triple the constraint operator contains three terms labelled by the three possible pairs of edges that one can form from e,e',e''. Let us look at one of them, say (neglecting numerical factors)

$$tr((h_{\alpha(v;e,e')}-h_{(\alpha(v;e,e'))^{-1}})h_{s''}[h^{-1}_{s''},\hat{V}(U(v))])T_{\gamma,\vec{\pi},\vec{I}} \qquad (125)$$

where s,s',s'' are the segments of e,e',e'' incident at v that end in the endpoints of the three arcs $a(v;e,e')$ etc., $\alpha(v;e,e')$ is the loop $s\circ a(v;e,e')\circ(s')^{-1}$ and $U(v)$ is any system of mutually disjoint neighborhoods, one for each vertex v. Let j,j',j'' be the spins of edges e,e',e''. First it is easy to see that $h_{s''}[h^{-1}_{s''},\hat{V}(U(v))]$ is gauge invariant at the endpoint p'' of s'', thus the state (125) is also invariant at p'' and since p'' is a divalent vertex this is only possible if the segments s'' and $e''-s''$ of e'' carry the same spin in the decomposition of (125) into spin network functions. But $e''-s''$ carries spin j'' (no holonomy along $e''-s''$ appears), so we conclude that the spin of e'' is unchanged in the decomposition. However the same is not true for e and e': the term $h_{\alpha(v;e,e')}-h_{(\alpha(v;e,e'))^{-1}}$ is a multiplication operator and it raises the spin of $a(v;e,e')$ from zero to $1/2$. So in general the state (125) decomposes into four spin network state where the spins of the segments s,s' are raised or lowered by $1/2$, that is, they are $j\pm1/2, j'\pm1/2$ respectively while the spins of the segments $e-s,e'-s'$ are unchanged, namely j,j'.

Now let us look at the remaining piece of the Lorentzian part. It contains two factors of $\hat{K}$, which is proportional to $[\hat{V}(\sigma), H_E(1)]$. Now it was shown in [32] that only the term $[\hat{V}(U(v)), H_E(U(v))]$ survives corresponding to the vertex v. Since the volume operator does not change the graph or the labels, what remain is basically two successive actions of $\hat{H}_E$.
In summary, the Hamiltonian constraint has an action similar to a fourth-order consisting of creation and annihilation operators. What is being created and annihilated are the spins of the edges of the graph.
One more important observation leads us to the solution of the constraint, namely that $\hat{H}_E$ creates edges of a special kind (from now referred as *extraordinary edges*), the arcs $a(v; e, e')$. What is special about them is that they end in planar vertices which are either bi- or trivalent. However not only the edges, the labels they carry are special, since it is always 1/2.
This leads us to classify the full set of labels $\mathcal{S}$ of the spin networks - these are called *spin nets* and consist of the graph and its labels. Denote by $\mathcal{S}_0 \subset \mathcal{S}$, called sources, the set of spin nets which do not have extraordinary edges. From the sources one can construct so called level n spin nets $\mathcal{S}_n$ recursively as follows: let $s_0 \in \mathcal{S}_0$ be a spin net and define $\mathcal{S}_0(s_0) := \{s_0\}$. We obtain $\mathcal{S}_{n+1}(s_0)$ from $\mathcal{S}_n(s_0)$ by constructing spin network functions from spin nets of $\mathcal{S}_n(s_0)$, decomposing $\hat{H}_E T$ into spin networks and putting the corresponding spin nets into $\mathcal{S}_{n+1}(s_0)$. After this one can show that the following statements are true:
1) $\mathcal{S}_n(s_0^1)$ and $\mathcal{S}_m(s_0^2)$ are disjoint unless $n = m$ and $s_0^1 = s_0^2$.
2) The complement $\bar{\mathcal{S}} := \mathcal{S} - \mathcal{S}_0$ coincides with the set of spin nets of level greater than zero.
3) For each $s \in \mathcal{S}$ there is a unique integer n and source s_0 such that $s \in \mathcal{S}_n(s_0)$.

These properties enable us to write an appropriate ansatz for a solution to $\hat{H}_E$. To do this we must recall that the solution of the Hamiltonian constraint is a diffeomorphism invariant distribution, so we define first $[\mathcal{S}_n(s_0)] := \{[s]\}_{s \in \mathcal{S}_n(s_0)}$ where $[s]$ is the label for a diffeomorphism invariant state $T_{[s]}$. Now according to the program of RAQ for the solution of the constraint we must search for a distribution Ψ such that

$$< \Psi | \hat{H}_E | \phi >= 0$$

for all ϕ. Of course this is equivalent with the expression where $\phi := T_{[s]}$ since the spin network functions form a complete basis. Now let the ansatz be the following:

$$\Psi := \Psi_{s_0,n} = \sum_{k=1}^{N} \sum_{[s]\in\mathcal{S}_{n_k}(s_0)} c_{[s]} T_{[s]} \tag{126}$$

where $c_{[s]}$ are the (complex) coefficients determined from the constraint equation. If we substitute this into it we obtain the condition

$$0 =< \Psi|\hat{H}_E(N)|T_{[s]} >= \sum_{k=1}^{N} \sum_{[s']\in\mathcal{S}_{n_k}(s_0)} c_{[s']} \sum_{v\in V(\gamma)} N(v) < T_{[s']}|\hat{H}_E(v)|T_{[s]} > . \tag{127}$$

Now it is clear that this expression can be non-vanishing if $s \in \mathcal{S}_{n_k-1}(s_0)$ for some k, say k=l. And because (127) has to be true for all $N(v)$ we obtain the following condition on the coefficients:

$$\sum_{[s']\in\mathcal{S}_{n_l}(s_0)} c_{[s']} < T_{[s']}|\hat{H}_E(v)|T_{[s]} >= 0. \tag{128}$$

This should vanish for all v and finite number of $[s] \in \mathcal{S}_{n_l-1}(s_0)$. This means that we obtained a finite system of linear equations on the coefficients $c_{[s']}$. Since the cardinality of the sets $\mathcal{S}_n$ grow exponentially with n the system is far from being over-determined and we arrive at an infinite number of solutions.

The method for solving the Lorentzian part is similar, but it becomes complicated since the coefficients from different levels get coupled and one gets solutions labelled by the highest level that was used.

4.5 Quantizing the matter Hamiltonian

The quantization of the matter contributions will be quite similar to the gravitational case. We will use the same method to eliminate the singular $1/\sqrt{q}$ from the constraints and to regulate the expressions. The differences will depend on the properties of the matter fields - for example in the case of the scalar field, eliminating the divergences in the kinetic term needs more care.

4.5.1 Yang-Mills sector

We begin with the electric piece $H_{EM,E} = \frac{1}{2Q^2}\int d^3x \frac{q_{ab}}{\sqrt{q}} E^a E^b$. Now let $\chi_\epsilon(x-y) := \prod_{i=1}^3 \theta(\epsilon/2 - |x-y|)$ be the characteristic function of a cube with volume ϵ^3. Let us do a point-splitting (we lose diffeomorphism invariance during regularisation, but it will be recovered in the end) and use the identity $\frac{1}{\kappa}\{A_a^I, V\} = 2sgn(det(e))e_a^I$. We get

$$\begin{aligned} H_{EM,E}(N) &= \frac{1}{2Q^2\kappa^2}\lim_{\epsilon\to 0}\frac{1}{\epsilon^3}\int d^3x N(x)\frac{\{A_a^I(x), V(x)\}}{\sqrt[4]{det(q)(x)}}\underline{E}^a(x)\times \\ &\times \int d^3y \chi_\epsilon(x-y)\frac{\{A_a^I(y), V(y)\}}{\sqrt[4]{det(q)(y)}}\underline{E}^b(y) = \\ &= \frac{1}{2Q^2\kappa^2}\lim_{\epsilon\to 0}\int d^3x N(x)\{A_a^I(x), \sqrt{V(x,\epsilon)}\}\underline{E}^a(x)\times \\ &\times \int d^3y \{A_b^I(y), \sqrt{V(y,\epsilon)}\}\underline{E}^b(y) \end{aligned} \tag{129}$$

where we used that if $V(x,\epsilon) = \int d^3y \chi_\epsilon(x-y)\sqrt{det(q)}$ is the volume of the box then $\lim_{\epsilon\to 0}\frac{V(x,\epsilon)}{\epsilon^3} = \sqrt{det(q)}$. In other words, by keeping the constraint to be of density weight one we were able to remove the divergent $1/\sqrt{det(q)}$ factor.
A completely similar analysis can be done for the magnetic parts, resulting in the following expression:

$$\begin{aligned} &H_{EM,B}(N) = \\ &= \frac{1}{2Q^2\kappa^2}\lim_{\epsilon\to 0}\int d^3x N(x)\{A_a^I(x), \sqrt{V(x,\epsilon)}\}\underline{B}^a(x)\int d^3y \{A_b^I(y), \sqrt{V(y,\epsilon)}\}\underline{B}^b(y) \end{aligned} \tag{130}$$

To continue we follow the lines of the previous section, the regularisation of the gravitational Hamiltonian. First we triangulate σ with tetrahedra Δ and rewrite the integral $\int d^3x$ in (129) as $\sum_\Delta \int_\Delta$. For small Δ we can write that

$$\begin{aligned} &N(v)\epsilon_{JKL}\underline{E}(F_{JK})tr(\tau^I h_{s_L(\Delta)}\{h_{s_L(\Delta)}^{-1}, V(U(v))\}) \approx \\ \approx\ & -N(v)\frac{1}{2}\epsilon_{JKL}\epsilon_{abc}s_J^b(\Delta)s_K^c(\Delta)\underline{E}^a s_L^d(\Delta)\{A_d^I(s_L(0)), V(U(v))\} = \\ =\ & -3N(v)vol(\Delta)\underline{E}^a\{A_a^I(s_L(0)), V(U(v))\} \end{aligned} \tag{131}$$

where s are the segments of the tetrahedra based at v and $\underline{E}(F_{JK}) \approx \frac{1}{2}\epsilon_{abc}s^b_J(\Delta)s^c_K(\Delta)\underline{E}^a$ is the electric flux (F_{JK} is the surface parallel to the face determined by $s_J(\Delta), s_K(\Delta)$). Now we replace all quantities with their operator counterparts and adept the triangulation to the graph γ corresponding to the state acted upon. Following the lines of the previous section we obtain

$$\hat{H}_{EM,E}(N)f_\gamma = -\frac{m_P}{2\alpha_Q l_P^3}\sum_{v\in V(\gamma)} N(v)\left(\frac{1}{3}\frac{8}{E(c)}\right)^2$$
$$\sum_{v(\Delta)=v(\Delta')=v} tr(\tau^I h_{s_L(\Delta)}\{h^{-1}_{s_L(\Delta)}, V(U(v))\})\epsilon_{JKL}\hat{\underline{E}}(F_{JK}) \times$$
$$\times tr(\tau^I h_{s_P(\Delta)}\{h^{-1}_{s_P(\Delta)}, V(U(v))\})\epsilon_{MNP}\hat{\underline{E}}(F_{MN})f_\gamma \qquad (132)$$

Here $E(v) = n(v)(n(v)-1)(n(v)-2)/6$ is the combinatorial factor already familiar from section 4.4 ($n(v)$ is the valance of the vertex).
For the magnetic part one uses the following:

$$\underline{h}_{\alpha_{JK}} - 1 \approx -i\int_0^1 dt\dot{\alpha}_{JK}\underline{A}_a(s(t)) = -i\int_{F_{JK}} \underline{B}_a dS^a$$

Thus the corresponding operator of the magnetic part is

$$\begin{aligned}\hat{H}_{EM,B}(N)f_\gamma &= -\frac{m_P}{2\alpha_Q l_P^3}\sum_{v\in V(\gamma)} N(v)\left(\frac{1}{3}\frac{8}{E(c)}\right)^2 \times \\ &\times \sum_{v(\Delta)=v(\Delta')=v} tr(\tau^I h_{s_L(\Delta)}\{h^{-1}_{s_L(\Delta)}, V(U(v))\})\epsilon_{JKL}(\hat{\underline{h}}_{\alpha_{KL}} - 1)) \times \\ &\times\ tr(\tau^I h_{s_P(\Delta)}\{h^{-1}_{s_P(\Delta)}, V(U(v))\})\epsilon_{MNP}(\hat{\underline{h}})_{\alpha_{NP}} - 1)f_\gamma\end{aligned} \qquad (133)$$

4.5.2 Scalar field

First let us focus on the kinetic term. The term proportional to p^2 looks hopelessly divergent, since we cannot absorb the factor $\sqrt{det(q)}$ anywhere (there are no gravitational fields beside this in the expression). To solve this problem let us insert the factor $1 = det(e^I_a)^2/\sqrt{det(q)}$ into the kinetic term and consider the following four-fold point-splitting:

$$H^\epsilon_{SK,kin}(N) =$$

$$
\begin{aligned}
&= \frac{1}{2}\int d^3xN(x)p(x)\int d^3yp(y)\int d^3u\frac{det(e^I_a(u))}{\sqrt{V(\epsilon,u)^3}}\int d^3v\frac{det(e^I_a(v))}{\sqrt{V(\epsilon,v)^3}}\times \\
&\times\ \chi_\epsilon(x,y)\chi_\epsilon(u,x)\chi_\epsilon(v,y) = \frac{1}{2}\frac{(-2)^2}{(3!)^2\kappa^6}\int d^3xN(x)p(x)\int d^3yp(y)\times \\
&\times\ \int d^3utr(\{A(u),V(u,\epsilon)\}\wedge\{A(u),V(u,\epsilon)\}\wedge\{A(u),V(u,\epsilon)\})\times \\
&\times\ \int d^3vtr(\{A(v),V(v,\epsilon)\}\wedge\{A(v),V(v,\epsilon)\}\wedge\{A(v),V(v,\epsilon)\}) \qquad (134)
\end{aligned}
$$

where we used the identity $\int d^3xdet(e^I_a) = (1/3!)\int \epsilon_{IJK}e^J\wedge e^J\wedge e^K = -(1/3)\int tr(e\wedge e\wedge e)$. So with this trick we were able to remove the problematic $1/\sqrt{det(q)}$ term from the denominator. Adapting the triangulation to the graph γ and using the identity

$$
\begin{aligned}
&\int_\Delta tr(\{A(x),V(x,\epsilon)\}\wedge\{A(x),V(x,\epsilon)\}\wedge\{A(x),V(x,\epsilon)\})\approx \\
\approx\ & \frac{1}{6}\epsilon_{IJK}tr(h_{s_I(\Delta)}\{h^{-1}_{s_I(\Delta)},\sqrt{V(v(\Delta))}\})tr(h_{s_J(\Delta)}\{h^{-1}_{s_J(\Delta)},\sqrt{V(v(\Delta))}\})\times \\
\times\ & tr(h_{s_K(\Delta)}\{h^{-1}_{s_K(\Delta)},\sqrt{V(v(\Delta))}\}) \qquad (135)
\end{aligned}
$$

and by replacing the fields with their operator counterparts and the Poisson brackets by commutators times $1/(i\hbar)$ we obtain:

$$
\begin{aligned}
&\hat{H}_{SK,kin}(N)f_\gamma = -\frac{1}{18\cdot 36 l_P^3}\sum_{v\in V(\gamma)}N(v)\hat{\Pi}^2(v)\left(\frac{8}{E(v)}\right)^2\sum_{v(\Delta)=v'(\Delta)=v}\times \\
\times\ & \epsilon_{ijk}tr(h_{s_i(\Delta)}\{h^{-1}_{s_i(\Delta)},\sqrt{V(v)}\})tr(h_{s_j(\Delta)}\{h^{-1}_{s_j(\Delta)},\sqrt{V(v)}\})tr(h_{s_k(\Delta)}\{h^{-1}_{s_k(\Delta)},\sqrt{V(v)}\})\times \\
\times\ & \epsilon_{lmn}tr(h_{s_l(\Delta)}\{h^{-1}_{s_l(\Delta)},\sqrt{V(v)}\})tr(h_{s_m(\Delta)}\{h^{-1}_{s_m(\Delta)},\sqrt{V(v)}\})tr(h_{s_n(\Delta)}\{h^{-1}_{s_n(\Delta)},\sqrt{V(v)}\})f_\gamma
\end{aligned}
\qquad (136)
$$

This operator is quite complicated, but it is well-defined.
Next we turn to the derivative term. We write

$$
q^{ab}\sqrt{det(q)} = \frac{E^a_I E^b_I}{\sqrt{det(q)}} \text{ and } E^a_I = \epsilon^{acd}\epsilon_{IJK}\frac{e^J_c e^K_d}{2}
$$

and regulate:

$$
H^\epsilon_{SK,der}(N) =
$$

$$
\begin{aligned}
&= \frac{1}{2}\int d^3x \int d^3y N(X)\chi_\epsilon(x,y)\epsilon^{IJK}\epsilon IMN\epsilon^{abc}\frac{(\partial_b\Phi e_b^J e_c^K)(x)}{\sqrt{V(x,\epsilon)}}\epsilon^{def}\frac{(\partial_d\Phi e_e^J e_f^K)(y)}{\sqrt{V(y,\epsilon)}} = \\
&= \frac{1}{2\kappa^4}\left(\frac{2}{3}\right)^4 \int d^3x N(x)\epsilon^{IJK}\partial\Phi(x)\wedge\{A^J(x),V(x,\epsilon)^{3/4}\}\wedge\{A^K(x),V(x,\epsilon)^{3/4}\}\times \\
&\times \int d^3y\chi_\epsilon(x,y)\epsilon^{IMN}\partial\Phi(y)\wedge\{A^M(y),V(y,\epsilon)^{3/4}\}\wedge\{A^N(y),V(y,\epsilon)^{3/4}\}
\end{aligned}
\tag{137}
$$

To rewrite this in terms of holonomies and point-holonomies we use the identities

$$
U(v)^{-1}U(s(\delta t)) - 1 = \exp(-i\Phi(v) + i\Phi(v) + i\delta t\dot{s}^a(0)\partial_a\Phi(v)) - 1 \approx i\delta t\dot{s}^a(0)\partial_a\Phi(v)
$$

(where s is some segment $(s(0)=v)$) and

$$
\begin{aligned}
& 6\int_\Delta d^3y\partial\Phi(y)\wedge\{A^J(y),V(y,\epsilon)^{3/4}\}\wedge\{A^K(y),V(y,\epsilon)^{3/4}\}\approx \\
\approx\ & -\epsilon^{mnp}(U(s_m(\Delta))U^{-1}(v)-1)tr(\tau_J h_{s_n(\Delta)}\{h^{-1}_{s_n(\Delta)},V(v(\Delta),\epsilon)\})\times \\
\times\ & tr(\tau_K h_{s_p(\Delta)}\{h^{-1}_{s_p(\Delta)},V(v(\Delta),\epsilon)\}).
\end{aligned}
\tag{138}
$$

Then the action of the operator corresponding to the derivative term on a cylindrical function is

$$
\begin{aligned}
& \hat{H}_{SK,der}f_\gamma = \left(\frac{2}{3}\right)^6\frac{\kappa m_P}{2l_P^9}\sum_{v\in V(\gamma)}N(v)\epsilon^{ijk}\epsilon^{ilm}\times \\
\times\ & \sum_{v(\Delta)=v(\Delta')=v}\left(\frac{8}{E(v)}\right)^2\epsilon^{npq}\epsilon^{rst}U(v)^{-1}U(s_n(\Delta))U(v)^{-1}U(s_r(\Delta'))\times \\
\times\ & tr(\tau_j h_{s_p(\Delta)}[h^{-1}_{s_p(\Delta)},V(v(\hat\Delta))^{3/4}])tr(\tau_k h_{s_q(\Delta)}[h^{-1}_{s_q(\Delta)},V(v(\hat\Delta))^{3/4}])\times \\
\times\ & tr(\tau_l h_{s_s(\Delta)}[h^{-1}_{s_s(\Delta)},V(v(\hat\Delta))^{3/4}])tr(\tau_m h_{s_t(\Delta)}[h^{-1}_{s_t(\Delta)},V(v(\hat\Delta))^{3/4}])f_\gamma
\end{aligned}
\tag{139}
$$

Again, despite its complicated appearance, this is a well-defined operator.
The potential term is trivial to quantize. The only question is how one can express $V[\Phi]$ as a function of point-holonomies. If we can, then the operator corresponding to $V[\Phi]$ will be

$$
\hat{V}[\Phi]f_\gamma = \hbar\sum_{v\in V(\gamma)}N(v)V[U(v)]\hat{V}(v)f_\gamma. \tag{140}
$$

5 Proca-field

In this section we will investigate the Proca-field in the framework of Loop Quantum Gravity. It turns out that the methods mentioned in the previous sections can be applied to the symplectically embedded Proca-field, giving a rigorous, consistent, non-perturbative quantization of the theory. This is not a trivial issue since the analysis of the 3+1 decomposition yields a second class constraint algebra, which is problematic if one wants to adapt the techniques of RAQ. The theory is also not gauge invariant, but it turns out that these two problems are related. The way out of this is a method called symplectical embedding [29] which states that by introducing a new field into the theory one obtains an equivalent theory that will have a first class constraint algebra. This also makes the theory gauge invariant. This section is dedicated to examine the questions arising from this method:
How can we obtain the original theory from the symplectically embedded one at the classical level (the question of gauge fixing)?
Can this be done in the quantum theory?
Are the degrees of freedom differ in the two cases?
What can we say about the $m \to 0$ case?
What is the role of the scalar field?

5.1 Classical theory

The action of the Proca-field coupled to gravity has the form

$$S = \int d^4x\mathcal{L}$$
$$\mathcal{L} = \mathcal{L}_{GR} + \mathcal{L}_{EM} - \frac{1}{2}m^2 g_{\mu\nu}\underline{A}^\mu\underline{A}^\nu, \tag{141}$$

where $\mathcal{L}_{GR}$ and $\mathcal{L}_{EM}$ are the Lagrangian density of the gravitational and electromagnetic fields.
To apply the framework of loop quantum gravity to this system, first we have to do a 3+1 decomposition. Using the notations of the second section we obtain for the Hamiltonian of

the Proca-field

$$H = \int_\sigma \Big(N\mathcal{H}^{GR+EM} + N^a \mathcal{H}_a^{GR+EM} + A_0^i G_i + A_0 \underline{G} + \\ + \sqrt{q}\frac{m^2}{2}\big(-\frac{A_0^2}{N^2} + \frac{2}{N^2}A_0 N^b \underline{A}_b - \frac{1}{N^2}(N^b \underline{A}_b)^2\big)\Big) \tag{142}$$

where $\mathcal{H}^{GR+EM}, \mathcal{H}_a^{GR+EM}, G_i$ and $\underline{G}$ are the contribution of the gravitational and electromagnetic fields to the scalar and diffeomorphism constraints and the gravitational and electromagnetic gauge constraints respectively. Since $\dot{\Pi}_0 = 0$ etc. must hold, we get the following consistency conditions (secondary constraints):

$$0 = \{\Pi_N, H\} = \mathcal{H}^{GR+EM} + \frac{1}{2}\sqrt{q}m^2\tilde{A}^2 \quad := \quad \tilde{\mathcal{H}} \tag{143}$$

$$0 = \{\Pi_a, H\} = \mathcal{H}_a^{GR+EM} + \sqrt{q}m^2\tilde{A}\underline{A}_a \quad := \quad \tilde{\mathcal{H}}_a \tag{144}$$

$$0 = \{\Pi_0, H\} = \underline{G} - \sqrt{q}m^2\tilde{A} \quad := \quad \underline{\tilde{G}} \tag{145}$$

$$0 = \{\Pi_0^i, H\} \quad = \quad G_i, \tag{146}$$

where we used the notation $\tilde{A} = \frac{A_0 - N^a \underline{A}_a}{N}$. One can verify that the above constraints have a second class constraint algebra. To show this, first let us introduce the following linear combination of the primary constraints:

$$\tilde{\Pi}_0 \quad := \quad \Pi_N + \tilde{A}\Pi_0 \tag{147}$$

$$\tilde{\Pi}_N \quad := \quad \Pi_N + N^a\Pi_a + A_0\Pi_0 \tag{148}$$

$$\tilde{\Pi}_a \quad := \quad \Pi_a + \underline{A}_a\Pi_0 \tag{149}$$

It is easy to show that the above linear combinations have weakly vanishing Poisson-brackets with all constraints, so these are first class constraints. The constraints G_i (gravitational Gauss constraint) and $\tilde{\mathcal{H}}_a + \underline{A}_a\underline{\tilde{G}} + A_a^i G_i$ (diffeomorphism constraint) are also first class. There are only two second class constraints: $\underline{\tilde{G}}$ and $\tilde{\mathcal{H}}$. Their Poisson-bracket is

$$\begin{aligned} & \{\tilde{\mathcal{H}}(x), \underline{\tilde{G}}(y)\} = \\ = \quad & m^2 \frac{\underline{E}^a(x) N_a(x)}{N(y)} \frac{\sqrt{q}(y)}{\sqrt{q}(x)}\delta(x-y) - \sqrt{q(x)}m^2\Big(\underline{A}^a(x) - \frac{\tilde{A}(x)N^a(x)}{N(x)}\Big)\mathcal{D}_a^{(y)}\delta(x-y) := \\ := \quad & M(x,y), \end{aligned} \tag{150}$$

where $\mathcal{D}_a^{(y)}$ means that the derivative should be calculated in the y variable.
To deal with second class systems, one needs to introduce the so called Dirac-brackets instead of the Poisson brackets. In the case of field theories, it is done in the following way (see [26] for details): first one calculates the matrix $M_{ij}(x,y) := \{C_i(x), C_j(y)\}$, where $C_i(x)$ are the second class constraints in the theory. After that one calculates the inverse of $M_{ij}(x,y)$ in the following sense (since $M_{ij}(x,y)$ is a distribution):

$$\int d^3z M_{ik}(x,z)(M^{(-1)})_{kj}(z,y) = \delta_{ij}\delta(x-y) \tag{151}$$

After this the Dirac-bracket is defined as

$$\{f,g\}_D := \{f,g\} - \int d^3x d^3y \{f, C_i(x)\}(M^{(-1)})_{ij}(x,y)\{C_j(y), g\} \tag{152}$$

In our case, $M_{ij}(x,y)$ is a 2 by 2 matrix with components

$$\begin{aligned} M_{11}(x,y) &= M_{22}(x,y) = 0 \\ M_{12}(x,y) &= -M_{21}(y,x) = M(x,y) \end{aligned} \tag{153}$$

The inverse of this matrix has the same structure:

$$\begin{aligned} (M^{(-1)})_{11}(x,y) &= (M^{(-1)})_{22}(x,y) = 0 \\ (M^{(-1)})_{12}(x,y) &= -(M^{(-1)})_{21}(y,x) = \tilde{M}(x,y), \end{aligned} \tag{154}$$

where $\tilde{M}(x,y)$ satisfies the following differential equation

$$m^2 \frac{\underline{E}^a(x)N_a(x)}{N(y)}\sqrt{q}(x)\tilde{M}(x,y) - \sqrt{q(x)}m^2\Big(\underline{A}^a(x) - \frac{\tilde{A}(x)N^a(x)}{N(x)}\Big)\mathcal{D}_a^{(x)}\tilde{M}(x,y) = \delta(x-y), \tag{155}$$

thus the Dirac-bracket has the form

$$\{f,g\}_D := \{f,g\} + \int d^3x d^3y \Big(\{f,\tilde{\mathcal{H}}(x)\}\tilde{M}(y,x)\{\tilde{G}(y),g\} - \{f,\tilde{G}(x)\}\tilde{M}(x,y)\{\tilde{\mathcal{H}}(y),g\}\Big) \tag{156}$$

The above construction has two drawbacks: the first is that the above Lagrangian is not gauge invariant. The cause of this is the mass term, since if one replaces m=0, we will have a first class constraint algebra. This affects only the U(1) gauge, SU(2) symmetries are still valid (since the mass term contains gravitational variables in the form of the metric tensor and its determinant). The second problem is that one is lead to a system with second-class constraints. The latter problem is more troublesome because the canonical quantization becomes quite difficult due to the fact that it is far nontrivial to implement the Dirac-brackets in the quantum theory ([28]).

There is an elegant way of curing both problems ([28],[29],[30]), and that is to introduce an auxiliary scalar-field and modify the Lagrangian to have the following form:

$$\mathcal{L}_m = \mathcal{L}^{G+EM} - \frac{1}{2}m^2 g_{\mu\nu}(\underline{A}^\mu + \partial^\mu\phi)(\underline{A}^\nu + \partial^\nu\phi) = \mathcal{L}^{G+EM} + \mathcal{L}^M \tag{157}$$

Note that the above Lagrangian is gauge invariant if the transformation rule for the fields under gauge transformation is

$$\delta\underline{A}^4_\mu = \partial^4_\mu\Lambda \tag{158}$$

$$\delta\phi = \Lambda \tag{159}$$

and the original Lagrangian is obtained via the gauge-fixing $\partial^4_a\phi = 0$.

To see that the system is first class, we perform the 3+1 decomposition to the modified Lagrangian as well. Using the notations already introduced we obtain

$$\begin{aligned}
H_m &= \int_\Sigma (N\mathcal{H} + N^a\mathcal{H}_a + A^i_0 G_i + A_0\underline{G}) && (160)\\
\mathcal{H} &= \mathcal{H}^{G+EM} + \frac{\pi^2}{2\sqrt{q}m^2} + \frac{\sqrt{q}m^2}{2}q^{ab}(\underline{A}_a + \partial_a\phi)(\underline{A}_b + \partial_b\phi) && (161)\\
\mathcal{H}_a &= \mathcal{H}^{G+EM}_a + (\underline{A_a} + \partial_a\phi)\pi && (162)\\
\underline{G} &= \mathcal{D}_a\underline{E}^a - \pi && (163)\\
G_i &= \mathcal{D}_a E^a_i, && (164)
\end{aligned}$$

where π is the conjugate momenta for ϕ:

$$\pi = \frac{\delta S}{\delta\dot{\phi}} = \frac{\sqrt{q}m^2(A_0 - N^a A_a + \mathcal{L}_t\phi - N^a\partial_a\phi)}{N} \tag{165}$$

We also have the primary constraints $\Pi_N = \Pi_a = \Pi^i = \Pi_0 = 0$ which are the same as in the previous case. If we calculate $\dot{\Pi}_N$ etc., we find that $\mathcal{H}, \mathcal{H}_a, G_b, \underline{G}$ are secondary constraints

in the theory. These constraints are the Hamiltonian, the diffeomorphism (modulo gauge transformations), the gravitational Gauss and Maxwell Gauss-constraints.
Before we turn to the quantization, it is worth to observe some of the properties of the above system:

- It is easy to check that the above system is first class, i.e. the constraint algebra is closed. This is due to the fact that the canonical momenta of the scalar field appears in the Gauss constraint.

- Note that the mass only appears in the scalar constraint, which means that gauge- and diffeomorphism symmetries are independent of m.

- The Hamiltonian is a linear combination of constraints, which is not true for the case where there is no scalar field, since there the Hamiltonian is quadratic in the Lagrange-multipliers.

- The scalar field and the Yang-Mills field is only coupled to each other in the scalar constraint and only through a derivative term and no scalar mass-term required, which means that if we will quantize this system the scalar field will have a totally different role than the one introduced via symmetry breaking.

- The term $\underline{A}_a + \partial_a \phi$ is gauge-invariant with respect to the gauge transformations generated by (163), so this is different to the case when we couple a scalar field to a gauge field via covariant derivatives (actually its more like an affine field).

- One cannot replace $m = 0$ in the Hamiltonian formalism to obtain the usual Maxwell-field. This is not unfamiliar in loop quantum gravity, since it resembles to the case of the Immirzi parameter (by this analogy we do not mean any deeper connection, though). There the connection and the electric field can be rescaled as $A^i_a \to A^i_a + \beta K^i_a$, $E^i_a \to \frac{E^i_a}{\beta}$. This is a canonical transformation since the Poisson-brackets are invariant under this transformation. If we substitute the new quantities in the Hamiltonian, we find that the Gauss and diffeomorphism constraints are unchanged, but one part of the Hamiltonian constraint will have a term proportional with $\beta^2 + 1$. (see eg. [36], [37] and references therein). Now consider the following canonical transformation:

$$\pi \to m\pi \quad \phi \to \frac{\phi}{m}$$

$$\underline{E}_a \to m\underline{E}_a \quad \underline{A}_a \to \frac{\underline{A}_a}{m}$$

This will remove the m parameter from the mass term and furthermore this parameter will appear only in the Hamiltonian constraint, the other constraints will be independent of m.

- One may ask whether the introduction of an extra field into the theory results in the change of the degrees of freedom. In the symplectically embedded case we have 18(gravitational)+6(Maxwell)+2(scalar)=26 variables in phase space, and we have 8 first class constraints. Since each of those constraints remove two degrees of freedom, we will have 26-16=10 in phase space. In the original case, though we have 24 variables, we obtain 6 first class and two second class constraints, and because the latter removes only one degree of freedom, we obtain 24-12-2=10 independent variables. We can interpret this as the extra degrees are gauged out.

5.2 Quantization

Since our fields consist of the gravitational, the electromagnetic and the scalar field, and we have a first class constraint algebra, we may apply the tools of RAQ directly which were mentioned in the previous sections. If one wants to quantize the original case (where we had a second class constraint algebra), the momentum operators should be modified by replacing the Poisson-brackets with Dirac-brackets. Specifically the momentum operators should be redefined in the following way:

$$\begin{aligned} \hat{E}(S)_D f(A) &:= i\bar{h}\{E(S), f(A)\}_D = \\ &= i\bar{h}\{E(S), f(A)\} - \\ &- i\bar{h}\int d^3x d^3y \{E(S), C_i(x)\}(M^{(-1)})_{ij}(x,y)\{C_j(y), f(A)\}, \end{aligned} \tag{166}$$

where C_i are the second class constraints $\underline{G}$ and $\bar{H}$. This does not modify the properties of the momentum operators, since the Dirac-brackets have the same properties as the Poisson-brackets. Only the action on spin network functions changes, which becomes more complicated since $\tilde{M}(x,y)$ is not explicit and depend also on $\underline{A}_a$ and $\underline{E}_a$. In fact only the momentum operator of the Maxwell-field changes, since $\{\tilde{G}, A^i_a\} = \{\tilde{G}, E^i_a\} = 0$. The only question is whether the momentum operator defined above is a well defined operator on the Hilbert space, since it is not trivial if its action is a cylindrical function. The more detailed analysis of this question can be found in section 5.5.

5.3 The Hamiltonian of the Proca-field

The total (non-smeared) Hamiltonian of the symplectically embedded Proca-field has the form

$$\begin{aligned} H &= \int_\Sigma N(\mathcal{H}_{GR+EM} + \mathcal{H}_P + \mathcal{H}_M) \\ \mathcal{H}_P &= \frac{\pi^2}{2\sqrt{q}m^2} \\ \mathcal{H}_M &= \frac{\sqrt{q}m^2}{2} q^{ab}(\underline{A}_a + \partial_a\phi)(\underline{A}_b + \partial_b\phi), \end{aligned} \tag{167}$$

where $\mathcal{H}_{GR+EM}$ is the Hamiltonian density of the electromagnetic field coupled to gravity, H_P is the kinetic term and H_M is the mass term (in the following we consider only the symplectically embedded Hamiltonian). To arrive at a well defined, diffeomorphism covariant Hamiltonian operator, we will use the regularisation introduced in section 4. The kinetic term is the same:

$$\begin{aligned} \hat{H}_P = \frac{1}{18\cdot 36m^2}\sum_v N(v)\Big(8\frac{\hat{\Pi}(v)}{E(v)}\Big)^2 \sum_{v(\Delta)=v(\Delta')=v} \epsilon^{IJK}\epsilon^{LMN}\epsilon_{ijk}\epsilon_{lmn} \times \\ \times \hat{Q}^i_{s_I(\Delta)}(v,\frac{1}{2})\hat{Q}^j_{s_J(\Delta)}(v,\frac{1}{2})\hat{Q}^k_{s_K(\Delta)}(v,\frac{1}{2})\hat{Q}^l_{s_L(\Delta')}(v,\frac{1}{2})\hat{Q}^m_{s_M(\Delta')}(v,\frac{1}{2})\hat{Q}^n_{s_N(\Delta')}(v,\frac{1}{2}), \end{aligned} \tag{168}$$

where $\hat{Q}^k_e(v,r) = tr(\tau_k h_e[h_e^{-1}, V(v)^r])$, $E(v) = \frac{n(n-1)(n-2)}{6}$ (n is the valance of the vertex v) and F_{JK} is a surface parallel to the face determined by s_J and s_K.

The mass term will be a bit different, but it can be obtained from the derivative term of 4.5.2 with the replacement $\partial\phi \to \partial\phi + A$, so the calculation is completely similar. From $q^{ab}\sqrt{q} = \frac{E^a_i E^b_i}{\sqrt{q}}$ and $E^a_i = \epsilon^{acd}\epsilon_{ijk}\frac{e^j_c e^k_d}{2}$ we have

$$\begin{aligned} H_M &= \frac{m^2}{2}\int d^3x \int d^3y N(x)\chi_\epsilon(x,y)\epsilon^{ijk}\epsilon^{ilm}\epsilon_{abc}\epsilon_{bef}\frac{((\partial_a\phi + \underline{A}_a)e^j_b e^k_c)(x)}{\sqrt{V(x,\epsilon)}}\frac{((\partial_b\phi + \underline{A}_b)e^m_e e^n_f)(y)}{\sqrt{V(y,\epsilon)}} = \\ &= \frac{m^2}{2}\Big(\frac{2}{3\kappa}\Big)^4 \int N(x)\epsilon^{ijk}(\partial\phi + \underline{A})(x)\wedge\{A^j(x), V(x,\epsilon)^{3/4}\}\wedge\{A^k(x), V(x,\epsilon)^{3/4}\}\times \end{aligned}$$

$$\times \quad \int \chi_\epsilon(x,y)\epsilon^{imn}(\partial\phi + \underline{A})(y) \wedge \{A^m(y), V(y,\epsilon)^{3/4}\} \wedge \{A^n(y), V(y,\epsilon)^{3/4}\} \tag{169}$$

Now using the expressions

$$\begin{aligned} U(1,s(\delta t)) &= \exp[i(\phi(v) + \delta t\dot{s}^a\partial_a\phi(v) + o(\delta t^2))] \\ \underline{h}_s(0,\delta t) &= \exp[i(\delta t\dot{s}^a\underline{A}_a + o(\delta t^2))] \end{aligned} \tag{170}$$

we have that

$$U(1,s(\delta t))\underline{h}_s(0,\delta t)U(1,v)^{-1} - 1 = i\delta t\dot{s}^a(\partial_a\phi(v) + \underline{A}_a(v)) + o(\delta t^2), \tag{171}$$

so after substitution we obtain

$$\begin{aligned} \hat{H}_M &= \frac{m^2\kappa m_P}{2l_P^9}\left(\frac{2}{3}\right)^6 \sum_v N(v)\left(\frac{8}{E(v)}\right)^2 \sum_{v(\Delta)=v(\Delta`)=v} \epsilon^{ijk}\epsilon^{ilm}\epsilon_{npq}\epsilon_{rst} \times \\ &\times \quad [U(1,s_n(\Delta))\underline{h}_{s_n(\Delta)}U(1,v)^{-1} - 1][U(1,s_r(\Delta`))\underline{h}_{s_r(\Delta`)}U(1,v)^{-1} - 1] \times \\ &\times \quad \hat{Q}^j_{s_p(\Delta)}(v,\frac{3}{4})\hat{Q}^k_{s_q(\Delta)}(v,\frac{3}{4})\hat{Q}^l_{s_s(\Delta`)}(v,\frac{3}{4})\hat{Q}^m_{s_t(\Delta`)}(v,\frac{3}{4}) \end{aligned} \tag{172}$$

This part of the Hamiltonian looks problematic due to open ends of the holonomies, but since U(1) gauge invariance is studied with respect to (163) and $\{\underline{G}, \underline{A}_a + \partial_a\phi\} = 0$ the term $\hat{H}_M$ is U(1) gauge invariant (open ends of holonomies are compensated by the scalar field, since the latter is defined in the vertices). Further more this term is also diffeomorphism and SU(2) gauge invariant, thus during quantization we do not come up against any problems. The total Hamiltonian of the (symplectically embedded) Proca-field is

$$\hat{H} = \hat{H}_{G+EM} + \hat{H}_P + \hat{H}_M$$

As Thiemann noticed earlier for similar systems, the Hamiltonian is well-defined, i.e. it doesn't suffer from UV divergences, and this is achieved not via renormalization or spontaneous symmetry breaking but treating the gravitational field dynamical.

5.4 Kernel of the Hamiltonian of the Proca-field

5.4.1 Complete solution

Though this Hamiltonian is quite complicated, there are a lot of relevant informations that can be extracted from it. First let's look at the action of the different terms in the Hamiltonian on a generalized spin network state.

The action of the gravitational term $\hat{H}_G$ changes the graph (as pointed out in [36]) in a way that it adds additional edges (specifically extraordinary edges) to the graph and changes the intertwiners, but it doesn't affect the labels which correspond to the matter fields. The other parts of the Hamiltonian describe the matter fields. Their structure is similar: they all contain matter operators and the $\hat{Q}$ operator in some way which encode the interaction of the fields with gravity. This operator only changes the intertwiners, it doesn't change neither the colorings or the graph itself. The derivative operators - the electric part of the Yang-Mills and the kinetic term - don't change the graphs, only the coefficients, but the mass term and the magnetic term does.

It is not an obvious question whether this Hamiltonian possesses a non-trivial kernel, but we will show that the construction of generating a solution to the Hamiltonian constraint, which was introduced by Thiemann, can be generalized to the present case.

Let $|T>_{\gamma,\vec{\rho},\vec{l},\vec{m},\vec{n}} := |T>_s$ be a generalized spin network state. Then $<\Phi|$ is in the kernel of the Hamiltonian of the Proca-field if for all $|T>_s$ we have

$$< \Phi|\hat{H}|T>_s = 0 \tag{173}$$

The key observation of Thiemann was that the Hamiltonian of gravity acts as it generates so called extraordinary edges (see details in [36] or [37]). and with this the kernel can be constructed in the following way: Denote the set of labelled graphs (spin-nets) $S_0 \in (\gamma_0, l_0)$ which contain no extraordinary edges (these are the "sources"). Then compute S_{n+1} by acting $\hat{H}_G$ on the elements of S_n and decomposing them into spin-network states. The main advantage of the sets S_n that 1.) they are disjoint, i.e. $S_n \cap S_m = \delta_{mn}$ and 2.) finding a general diffeomorphism invariant solution to the Hamiltonian-constraint reduces to finding a solution on a finite subspace.

Since we are only interested in solutions which are diffeomorphism invariant, we use $T_{[s]}$ instead of T_s, where [s] labels the diffeomorphism invariant distribution. In particular, let the ansatz for a solution be of the form

$$< \Psi| := \sum_{i=1}^{N} \sum_{[s]\in[S^{n_i}]} c_{[s]} < T|_{[s]}$$

Since the Euclidean part of the gravitational Hamiltonian maps from $S^{(n)}$ to S^{n+1}, we have that the condition

$$\sum_{i=1}^{N} \sum_{[s]\in S^{n_i}} c_{[s]} < T|_{[s]} \hat{H}_{GE} |T >_{[s']} = 0 \tag{174}$$

is non-trivial if and only if $[s'] \in [S^{n_i - 1}]$. Since $\hat{H}_{GE} = \sum_v N(v) \hat{H}_{GE}(v)$ and the above equation has to hold for all possible N, we have

$$\sum_{i=1}^{N} \sum_{[s]\in S^{n_i}} c_{[s]} < T|_{[s]} \hat{H}_{GE}(v) |T >_{[s']} = 0 \tag{175}$$

for each choice of finite number of vertices v and spin nets. Thus, we arrived at a finite system of liner equations with finite number of coefficients.

In the case of the Proca-field we have three fields. Since the orthonormal base is of the form $|T > \otimes |F > \otimes |D >$, we will first look for analogs of the sets $S^{(n)}$ in the case of the scalar-field and the Yang-Mills field.

The case of the scalar field is simple: denote the set $S^{(0)}(U)(\gamma)$ of all colored graphs, that all vertices are labelled by zero. Now define $S^{(n+1)}(U)$ by acting with $U(1, v)$ (for all possible v) on every element of $S^{(n)}(U)$. From the simple action of $U(1, v)$ it is clear that this is equivalent to that the elements of $S^{(n)}(U)$ are those colored graphs, for which the sum of the vertex colorings are n. If we look at the form of $\hat{H}_M$ we find that it maps from $S^{(n)}(U)$ to $S^{(n)}(U) \cup S^{(n+1)}(U) \cup S^{(n+2)}(U)$.

The case of the Yang-Mills field is a bit more complicated, because both $\hat{H}_B$ and $\hat{H}_M$ changes the graph. The former adds two (Yang-Mills) loops with color 1, while the latter increases the color of two edges by one (non-existent edges can be treated like they were edges with coloring zero). This is why in this case the analogs of $S^{(n)}$ will have two indices. Denote by $S^{(n,m)}(YM)$ the set of labelled graphs which have n loops with color 1 and the sum of the colors on all edges are m. It is easy to see that these sets are disjoint and the action of the $\hat{H}_B$

and $\hat{H}_M$ operators are the following: while $\hat{H}_B$ maps from $S^{(n,m)}(YM)$ to $S^{(n+2,m+2)}(YM)$, $\hat{H}_M$ maps from $S^{(n,m)}(YM)$ to $S^{(n,m+2)}(YM) \cup S^{(n,m+1)}(YM) \cup S^{(n,m)}(YM)$. It follows from the construction that $S^{(0,0)}(YM)$ will contain labelled graphs with zero colorings on all edges. Also note that the set $S^{(n,m)}(YM)$ is empty unless $n \geq m$.

Now, let the ansatz for the solution to the kernel of the Proca-field be of the form

$$< \Psi| := \sum_{i,j,k,l} \sum_{[s] \in S^{(n_i)}} \sum_{[f] \in S^{(m_j,p_k)}(YM)} \sum_{[d] \in S^{(q_l)}(U)} c_{[s],[f],[d]} < T|_{[s]} \otimes < F|_{[f]} \otimes < D|_{[d]} \tag{176}$$

Now (176) is in the kernel of the Hamiltonian of the Proca-field, if for all [s]', [f]', [d]'

$$< \Psi|\hat{H}|T >_{[s]'} \otimes |F >_{[f]'} \otimes |D >_{[d]'} = 0 \tag{177}$$

With the same reasoning as before this condition is non-trivial if

-$[s]' \in [S^{n_i - 1}] \cup [S^{n_i - 2}]$ (the union of the two sets is necessary if one takes the action of $\hat{H}_{GL}$ also into account, since this latter operator adds two extraordinary edges)

-$[d]' \in [S^{(q_l)}(U)] \cup [S^{(q_l - 1)}(U)] \cup [S^{(q_l - 2)}(U)]$

-$[f]' \in [S^{(m_j - 2, p_k - 2)}(YM)] \cup [S^{(m_j, p_k - 2)}(YM)] \cup [S^{(m_j, p_k - 1)}(YM)] \cup [S^{(m_j, p_k)}(YM)]$

5.4.2 Special solutions

Since the system of equation (177) is very complicated, it is useful to take some special solutions in order to understand the full theory. Let $< 0|_{YM}$ be the "vacuum" flux network state of the Yang-Mills sector, which means that it has no Yang-Mills colors on either edge (note that this is not actually the familiar vacuum state as was shown in [43], since we are not dealing with Fock-spaces). Similarly, denote the vacuum vertex function and vacuum spin network state by $< 0|_U$ and $< 0|_G$ respectively. Now it is easy to check that the state $< \Psi|_G \otimes < 0|_{YM} \otimes 0|_U$ is a solution of (177) if $< \Psi_G|$ is the solution of the gravitational part of the Hamiltonian (this is because these "vacuum states" are annihilated by the corresponding derivative operators in $\hat{H}_P$ and $\hat{H}_{E_{YM}}$, and are orthogonal to every state created by the operators in $\hat{H}_{YM}$ and $\hat{H}_U$). So these special states can be interpreted as pure gravity.

Now let us check states of the form $< 0|_G \otimes < \Psi|_{YM} \otimes < 0|_U$. It is easy to show that this is in the kernel of the Hamiltonian for all $< \Psi|_{YM}$! In fact the same is true for states of the form $< 0|_G \otimes < \Psi|_{YM} \otimes < \Psi|_U$. These states are obviously nonphysical since the expectation value of the volume, area and length operators of these states are all zero for all volumes, surfaces and curves respectively. (It is worth mentioning that these states are in the kernel of all Hamiltonian which have density weight one and composed only from the gravitational, Yang-Mills and scalar fields)

Let us check whether there are solutions of the form $<\Psi|_G \otimes <\Psi|_{YM} \otimes <0|_U$, where $<\Psi|_G \otimes <\Psi|_{YM}$ is in the kernel of $\hat{H}_G + \hat{H}_{YM}$. The answer is yes, if $<\Psi|_G \otimes <\Psi|_{YM}$ contains only flux networks that have U(1) colors only on the loops, not on the edges, since in this case these states are annihilated by the operator $\hat{H}_M$. These states are in the subset of the kernel of the Yang-Mills field coupled to gravity. Since currently we do not have a semi-classical description of the above system it will be for future investigations to check the physical meaning of these states. But if we look at the limit $m \rightarrow 0$, we find that these states will be solutions, since $<\Psi|_G \otimes <\Psi|_{YM} \otimes <0|_U X(v)|\phi>= 0$ for all $|\phi>$ and

$$\begin{aligned} &<\Psi|_G \otimes <\Psi|_{YM} \otimes <0|_U (\hat{H} - \hat{H}_G - \hat{H}_{YM})|\phi>= \\ &\qquad =<\Psi|_G \otimes <\Psi|_{YM} \otimes <0|_U \hat{H}_M|\phi> \rightarrow 0 \end{aligned} \tag{178}$$

5.5 Gauge fixing

We used the symplectically embedded Proca-field to avoid implementing the Dirac-brackets in the quantum theory. This approach led to a well defined quantum theory as it was shown in the previous sections. This was achieved by introducing an auxiliary scalar field to the formalism. The theory we gained is equivalent to the original one since if the constraints are solved the scalar field disappears automatically. But this equivalence is not manifest if we do not solve the constraints, thus we need to introduce gauge fixing. As we shall see this will again lead to a system with second class constraints like in the original theory (without the scalar field). Below we outline how the quantization of such systems could be handled.

The key observation is that the original Lagrangian can be obtained via the gauge fixing $\mathcal{D}_\mu \phi = 0$. The strategy will be to implement this condition in the Hamiltonian formalism. If we compare the original Hamiltonian with the Hamiltonian of the symplectically embedded Proca-field we find that if we substitute

$$\partial_a \phi = 0 \tag{179}$$

$$\pi = \sqrt{q} m^2 \frac{A_0 - N^a \underline{A}_a}{N} \tag{180}$$

into (161)-(163), we obtain the constraints (143)-(145). So if we introduce the two extra conditions (constraints)

$$\begin{aligned} C_a &= \partial_a \phi = 0 && (181) \\ C &= \pi - \sqrt{q} m^2 \frac{A_0 - N^a \underline{A}_a}{N} = 0, && (182) \end{aligned}$$

we arrive to the original case. One may ask how come the original theory have second class constraints while the symplectically embedded one has only first class constraints. The trick is that, as it was pointed out in [48], the conditions (181) and (182) are also constraints and if we include these to the constraint algebra, we obtain a system with second class constraints. (In the case of second class constraints one must use Dirac-brackets, so it is natural to ask what did we gain with the symplectic embedding? Actually, the main advantage of this method that we were able to quantize the theory without introducing the Dirac-brackets. The Dirac-brackets are only needed when one fixes the gauge.)

Now what remains is to calculate the Dirac-bracket and implement the two conditions (181) and (182) in the quantum theory. First we need the Poisson brackets of the new constraints with the existing ones. If we define the same linear combinations (147),(148),(149) as for the original case, we find that these are first class constraints. Also with the same reasoning as we did there one finds that G_i and $\mathcal{H}_a + G_i A^i_a + \underline{G A}_a$ are also first class constraints. Thus we have six second class constraints: $\mathcal{H}, \underline{G}, C_a, C$. The elements of the antisymmetric matrix $M^{(P)}_{ij}(x,y)$ are therefor the following:

$$\begin{aligned} M^{(P)}_{12} &= \{\mathcal{H}(x), \underline{G}(y)\} = 0 && (183) \\ M^{(P)}_{13} &= \{\mathcal{H}(x), C_a(y)\} = \frac{\pi(x)}{m^2\sqrt{q}(x)} \mathcal{D}^{(y)}_a \delta(x-y) && (184) \\ M^{(P)}_{14} &= \{\mathcal{H}(x), C(y)\} = -m^2\sqrt{q}(x)(\underline{A}_a + \partial_a\phi)\mathcal{D}^{a(x)}\delta(x-y) + \\ &+ \; m^2 \frac{\sqrt{q}(y)}{\sqrt{q}(x)} \frac{N^a(y)\underline{E}_a(x)}{N(y)} \delta(x-y) && (185) \\ M^{(P)}_{23} &= \{\underline{G}(x), C_a(y)\} = -\mathcal{D}^{(y)}_a \delta(x-y) && (186) \\ M^{(P)}_{24} &= \{\underline{G}(x), C(y)\} = m^2 \frac{\sqrt{q}(y)}{N(y)} N^a(y) \mathcal{D}^{(x)}_a \delta(x-y) && (187) \\ M^{(P)}_{34} &= \{C_a(x), C(y)\} = -\mathcal{D}^{(x)}_a \delta(x-y) && (188) \end{aligned}$$

With the inverse matrix $(M^{(P)})^{(-1)}_{ij}(x,y)$ one can define the Dirac-brackets the similar way as in the first chapter, the only difference is that now we have six second class constraints and the matrix $M^{(P)}_{ij}(x,y)$ is much more complicated.
After this we quantize the theory in the following way. The Hilbert space and the configuration variables are defined as in the original (first class constraint) case, but we have to redefine the momentum operators $\underline{\hat{E}}(S)$ and $\hat{P}_B$ (see the beginning of section 5.2). Now let us check how our new constraints can be interpreted in the quantum theory. Because of the complicated Dirac-bracket, the precise action of the operator version of (182) is left for future studies. The constraint (181) on the other hand is much more simple. If we look at the regularisation of the mass term, specifically at equations (170), we see that

$$U(1,s(\delta t)) - U(1,v) = i\delta t\dot{s}^a\partial_a\phi(v) + o(\delta t^2), \tag{189}$$

so the constraint can be implemented as $<\Psi|$ is in its kernel only if

$$<\Psi|(U(1,b(e)) - U(1,f(e)))|\psi>= 0 \tag{190}$$

for all $|\psi>$ and all e edge (b(e) is the beginning-, f(e) is the endpoint of the edge). Let $|\psi>$ be a basis element, that is

$$|\psi>=|S>_{\gamma,\vec{j},\vec{\rho},\vec{l},\vec{\lambda}}=|T(A)>_{\gamma,\vec{j},\vec{\rho}}\otimes|F(\underline{A})>_{\gamma,\vec{l}}\otimes|D(U)>_{\gamma,\vec{\lambda}} \tag{191}$$

It is obvious that the action of the operator in (190) will be

$$(U(1,b(e)) - U(1,f(e)))|\psi>=|S>_{\gamma,\vec{j},\vec{\rho},\vec{l},\vec{\lambda}_1} - |S>_{\gamma,\vec{j},\vec{\rho},\vec{l},\vec{\lambda}_2}, \tag{192}$$

where λ_1 and λ_2 is obtained by increasing the value of λ_v in the appropriate vertex by one. Since the condition is implemented in all vertices, we have that the coefficient of $<S|_{\gamma,\vec{j},\vec{\rho},\vec{l},\vec{\lambda}}$ in $<\Psi|$ is the same as the coefficient of $<S|_{\gamma,\vec{j},\vec{\rho},\vec{l},\vec{\lambda}'}$ if both $|D(U)>_{\gamma,\vec{\lambda}}$ and $|D(U)>_{\gamma,\vec{\lambda}'}$ are elements of $S^{(n)}(U)$ for a fixed value of n. In other words if we substitute (176) into the above constraint we get the following condition on the coefficients

$$c_{[s],[f],[d]_1} - c_{[s],[f],[d]_2} = 0 \tag{193}$$

for all $[s],[f],[d]_1,[d]_2$, where $[d]_1,[d]_2$ are in the same set $S^{(n)}(U)$. This condition has non-trivial solutions, for example the case where only those coefficients of $< S|_{\gamma,\vec{j},\vec{\rho},\vec{l},\vec{\lambda}}$ are not zero where λ_v is the same for all vertices.

This was the case when one first implements gauge fixing, then quantize the system. One may ask whether it is possible to first quantize the system, then do gauge fixing. This is a problematic issue for the following reasons. Consider the operator versions of the constraints $\mathcal{H},\underline{G},C$. These, when quantized, are smeared with test functions N,A_0,Λ respectively and have the form $\hat{H}=\sum_v N(v)\hat{H}_v$ etc. In the case of C_a one simply uses the operator in (190). Now consider the operator matrix $\hat{M}_{ij}(v,v')$ - which could be interpreted as the operator version of $M^{(P)}_{ij}(x,y)$ - defined with the help of the commutators of the constraint operators: $\hat{M}_{12}=[\hat{H}_v,\hat{\underline{G}}_{v'}]$ etc. The question is whether the inverse of this matrix - defined via the condition $\sum_{v''}\hat{M}_{ik}(v,v'')(\hat{M})^{-1}_{kj}(v'',v')=\delta_{ij}\delta_{vv'}$ - actually exists. If it does, then with the help of this matrix we can define a Dirac commutator - analog of the Dirac-bracket - the following way:

$$
\begin{aligned}
&[\hat{O}_1,\hat{O}_2]_D=[\hat{O}_1,\hat{O}_2]-\\
&-\frac{1}{2}\sum_{v_1,v_2}([\hat{O}_1,\hat{C}_i(v_1)](\hat{M})^{-1}_{ij}(v_1,v_2)[\hat{C}_j(v_2),\hat{O}_2]-[\\
&-\hat{O}_2,\hat{C}_i(v_1)](\hat{M})^{-1}_{ij}(v_1,v_2)[\hat{C}_j(v_2),\hat{O}_1]),
\end{aligned}
\tag{194}
$$

where the $\hat{C}_i$ are the constraint operators. What now one has to do is to replace the commutators with this Dirac commutator. Further more one has to modify the action of the momentum operators. To see why, consider first a vertex function

$$
|D(U)>_{\gamma,\vec{\lambda}}:=\prod_{i=1}^{N}(U(\lambda_i,v_i))
\tag{195}
$$

which can be interpreted if the operator $\prod_{i=1}^{N}(U(\lambda_i,v_i))$ acted on the vacuum state $|0>$. If we have first class constraint algebra we have the following identity:

$$
\hat{P}|D(U)>_{\gamma,\vec{\lambda}}=[\hat{P},\prod_{i=1}^{N}(U(\lambda_i,v_i))]|0>
\tag{196}
$$

After gauge fixing this identity is the key to define the new momentum operator $\hat{P}_D$:

$$\hat{P}_D|D(U)>_{\gamma,\vec{\lambda}}:=[\hat{P},\prod_{i=1}^{N}(U(\lambda_i,v_i))]_D|0> \tag{197}$$

The new electric flux operator $\hat{E}(S)_D$ can be defined the same way. The only thing one has to do is replace the momentum operators in the constraints and use Dirac commutators instead of usual commutators (note that the gravitational momentum operator does not change since the Poisson-bracket of E^i_a is zero with all four second class constraints).
The critical part of the above construction is the existence and uniqueness of $(\hat{M})^{-1}_{ij}(v,v')$. But even if it would exist, there is the other question whether (194) is really the operator version of the Dirac-bracket? The answers to these questions are the requirement that the above construction works.
If the above operator exists then the anomalies of the constraint algebra are removed. First let us focus on the gravitational variables. Since the gravitational gauge and the diffeomorphism constraints are first class, and the momentum operator for the canonical momenta E^i_a does not change after gauge fixing, there will be no gravitational anomalies in the theory. Other anomalies will not appear since by construction $\{C_i,C_j\}_D=0$ for all constraints C_i and if (194) is the operator version of the Dirac-bracket, we obtain

$$\begin{aligned}
&[\hat{C}_i,\hat{C}_j]_D=[\hat{C}_i,\hat{C}_j]-\\
&-\frac{1}{2}\sum_{v_1,v_2}([\hat{C}_i,\hat{C}_k(v_1)](\hat{M})^{-1}_{kl}(v_1,v_2)[\hat{C}_l,\hat{C}_j]-[\hat{C}_j,\hat{C}_k(v_1)](\hat{M})^{-1}_{kl}(v_1,v_2)[\hat{C}_l,\hat{C}_i])=\\
&=[\hat{C}_i,\hat{C}_j]-\frac{1}{2}([\hat{C}_i,\hat{C}_j]-[\hat{C}_j,\hat{C}_i])=0
\end{aligned} \tag{198}$$

so there are no anomalies (the above commutator does not impose a new constraint since it is identically zero). This is also true for first class constraints that have vanishing Poisson-bracket with all constraints, since then the corresponding operators will have zero commutator, thus the previous expression is also zero. The only problem is the case of first class constraints that have non-zero Poisson-bracket with the constraints since then the structure constants will appear in the Dirac-bracket. This is problematic in the quantum theory since the structure constants may become operators which could cause anomalies. In all cases factor ordering ambiguities occur because we have terms that contain the product of three non-trivial operators, but these do not cause inconsistencies but only change the results of the theory.

5.6 Mass

Note that in this theory mass is a parameter, in fact a coupling constant which couples the scalar field, the Yang-Mills field and gravity. In the classical Hamiltonian analysis one can make the following rescaling: $\pi \to \pi/m$, $\phi \to m\phi$, $\underline{A}_a \to \underline{A}_a/m$, $\underline{E}_a \to m\underline{E}_a$, which is a canonical transformation. But in the quantum regime, this parameter enters the Hamiltonian in a non-trivial way. In this sense it is very similar to the Immirzi parameter of the pure gravitational case. In the latter case the Hawking-entropy provided a tool that helped fix this parameter ([36],[37]), so there is a chance that with a similar method one might be able to make predictions on the value of m.

Another way would be to define propagators in loop quantum gravity, since the poles of the propagators could be interpreted as mass. But so far the question of time remains unsolved in the theory, leaving this idea for future research. None the less there are attempts which could provide a solution of the problem of time, see e.g. [49] and references therein. But without further input, mass is an undefined parameter of the theory which has to be given from experiments.

6 Spontaneous symmetry breaking in Loop Quantum Gravity

In this section we investigate the question how spontaneous symmetry breaking works in the framework of Loop Quantum Gravity and we compare it to the results obtained in the case of the Proca field, where we were able to quantize the theory in Loop Quantum Gravity without introducing a Higgs field. We obtained that the Hamiltonian of the two systems are very similar, the only difference is an extra scalar field in the case of spontaneous symmetry breaking. This field can be identified as the field that carries the mass of the vector field. In the quantum regime this becomes a well defined operator, which turns out to be a self adjoint operator with continuous spectrum. To calculate the spectrum we used a new representation in the case of the scalar fields, which in addition enabled us to rewrite the constraint equations to a finite system of linear partial differential equations. This made it possible to solve part of the constraints explicitly.

6.1 Preliminaries

Currently spontaneous symmetry breaking is the most accepted tool to define mass to particles. Its success can be observed especially in the case of vector fields since their original Lagrangian - the Proca Lagrangian - is non-renormalizable. In the previous section we showed how one can quantize the massive vector field in Loop Quantum Gravity without spontaneous symmetry breaking. The main problem was that the Proca field had a second class constraint algebra which made it almost impossible to apply the framework of LQG. But with the help of symplectical embedding one could eliminate these difficulties. Now the question arises what is the difference between the two theories. To study this one first has to apply the framework of LQG to a system where spontaneous symmetry breaking is used to generate mass for a U(1) vector fiel. Next we will introduce a new basis for the scalar fields which is motivated by the fact that these are eigenstates of the configuration variables. It turns out that with the help of this new we are able to (partially) solve the constraints of the theory. We will also analyze special solutions in order to understand the role of the scalar field. In particular we find that some of these are almost identical to the solutions obtained for the Proca field, thus we are able to relate the two theories. Finally we wil turn our attention toward the “mass operator” and its properties, concentrating especially on those cases where the eigenvalues of this operator can be identified with the mass parameter of the Proca field. We obtain that in a sense the case of the symmetry breaking is a linear combination of infinite Proca theories.

6.2 Classical theory

First we will analyze the general framework of spontaneous symmetry breaking from a Hamiltonian perspective. We begin with the 3+1 decomposition of the theory original theory, while in the following we first have the U(1) field a VEV then perform the 3+1 decomposition (this is useful because the similarities between the Proca field and spontaneous symmetry breaking become more transparent).

6.2.1 Symmetric theory

For simplicity we use the Lagrangian of a U(1) vector field (electromagnetic field) coupled to gravity and a U(1) complex scalar field on a space-time manifold M. The Lagrangian of the scalar field is

$$L^{mat} = \int_M d^4x\sqrt{-g}\left(-\frac{1}{2}D_\mu\Phi^* D^\mu\Phi - \frac{1}{4}\mu(\Phi^*\Phi - a^2)^2\right), \tag{199}$$

where

$$\begin{aligned} \Phi &= \Re\Phi + i\Im\Phi \\ D_\mu\Phi &= (\partial_\mu + ie\underline{A}_\mu)\Phi \end{aligned}$$

and μ and a are positive constants. To distinguish between the electromagnetic field and the gravitational field the variables of the former are underlined.

First let us make the 3+1 decomposition. Using the notations introduced earlier we obtain

$$\begin{aligned} L^{mat} = \int dt \int d^3x \left(\frac{N}{\sqrt{q}} q_{ab} \frac{\underline{E}^a\underline{E}^b - \underline{B}^a\underline{B}^b}{2}\right) - \frac{1}{4}N\sqrt{q}\mu(\Phi^*\Phi - a^2)^2 - \\ -\frac{1}{2}N\sqrt{q}\left[q^{cd}D_c\Phi^* D_d\Phi - \frac{(\mathcal{L}_t\Phi^* - ieA^0\Phi^* - N^a(D_a\Phi)^*)(\mathcal{L}_t\Phi + ieA^0\Phi - N^bD_b\Phi)}{N^2}\right], \end{aligned} \tag{200}$$

where

$$\underline{E}_a = \frac{\sqrt{q}}{N}(\mathcal{L}_t\underline{A}_c - D_c\underline{A}^0 - \epsilon_{abc}\underline{B}^b N^c) \tag{201}$$

is the electric field and $\underline{B}_a$ is the magnetic field. We now define the canonical momenta

$$\begin{aligned}
\underline{\Pi}^a &= \frac{\delta L}{\delta\mathcal{L}_t\underline{A}_a} = \underline{E}^a && (202)\\
\pi &= \frac{\delta L}{\delta\mathcal{L}_t\Phi} = \sqrt{q}\frac{\mathcal{L}_t\Phi^* - ie\underline{A}^0\Phi^* - N^a(D_a\Phi)^*}{N} && (203)\\
\pi^* &= \frac{\delta L}{\delta\mathcal{L}_t\Phi^*} = \sqrt{q}\frac{\mathcal{L}_t\Phi + ie\underline{A}^0\Phi - N^b D_b\Phi}{N}. && (204)
\end{aligned}$$

Finally we perform the Legendre transformation to arrive to the Hamiltonian:

$$\begin{aligned}
H^{mat} &= \int d^3x(N\mathcal{H}^{mat} + N^a\mathcal{H}^{mat}_a + A_0\underline{G}) && (205)\\
\mathcal{H}^{mat} &= q_{ab}\left(\frac{\underline{E}^a\underline{E}^b + \underline{B}^a\underline{B}^b}{2\sqrt{q}}\right) + \frac{\pi\pi^*}{2\sqrt{q}} + \frac{1}{2}\sqrt{q}(\mathcal{D}_a\Phi)^*\mathcal{D}_a\Phi + \frac{1}{4}\sqrt{q}\mu(\Phi^*\Phi - a^2)^2 && (206)\\
\mathcal{H}^{mat}_a &= \epsilon_{abc}\underline{B}^c\underline{E}^b + \pi\mathcal{D}_a\Phi + \pi^*(\mathcal{D}_a\Phi)^* && (207)\\
\underline{G} &= D_a\underline{E}_a + ie(\pi^*\Phi^* - \pi\Phi) && (208)
\end{aligned}$$

where $\mathcal{H}^{mat}, \mathcal{H}^{mat}_a$ and $\underline{G}$ are the matter contributions to the Hamiltonian- and diffeomorphism (modulo gauge transformations) constraints, and the electromagnetic Gauss constraint.

The (non-smeared, non-trivial) Poisson-brackets are:

$$\begin{aligned}
\{\pi, \Phi\} &= \delta(x, y) && (209)\\
\{\pi^*, \Phi^*\} &= \delta(x, y) && (210)\\
\{E_a, A^b\} &= \delta^b_a\delta(x, y) && (211)
\end{aligned}$$

Since we are going to do symmetry breaking with the help of the scalar field, we write here the transformation rules for the scalar fields and their canonical momenta with respect to infinitesimal gauge transformation:

$$\{\underline{G}(\Lambda), \Phi\} = -ie\Lambda\Phi$$

$$
\begin{aligned}
\{\underline{G}(\Lambda), \Phi^*\} &= ie\Lambda\Phi^* \\
\{\underline{G}(\Lambda), \pi\} &= ie\Lambda\pi \\
\{\underline{G}(\Lambda), \pi^*\} &= -ie\Lambda\pi^*
\end{aligned}
$$

Before we continue, there are a few interesting observations that should be mentioned here:

- All the constraints are real and only the scalar fields and their canonical momenta are represented by complex variables (note that in the Hamiltonian picture Φ and Φ^* are independent variables).
- The transformation $\Phi \leftrightarrow \Phi^*,\ \pi \leftrightarrow \pi^*$ is a canonical transformation.
- The true diffeomorphism constraint $\mathcal{H}_a^{mat} + \underline{A}_a\underline{G}$ is independent of the coupling constant e (it contains partial derivatives only).
- This system has a first class constraint algebra, further more all the components of $\mathcal{H}_{mat}$ are gauge invariant respectively.

6.2.2 New variables

In spontaneous symmetry breaking first we introduce new fields η and Θ in the following way:

$$
\Phi(x) := (a + \eta(x)) \exp\left(i\frac{\Theta(x)}{a}\right). \tag{212}
$$

These variables are useful because the U(1) symmetry of the theory becomes more transparent. If we substitute this into the Lagrangian, we obtain

$$
\begin{aligned}
L_{mat} = \int d^4x\sqrt{-g}\Bigg[-\frac{1}{4}\underline{F}^{(4)}_{ab}\underline{F}^{(4)ab} - \frac{1}{2}\partial^{(4)}_a\eta\partial^{4)a}\eta - \\
-\frac{1}{2}(a+\eta)^2\left(\frac{\partial^{(4)}_a\Theta}{a} + e\underline{A}_a\right)\left(\frac{\partial^{(4)}_a\Theta}{a} + e\underline{A}_a\right) - \frac{1}{4}\mu\eta^2(2a+\eta)^2\Bigg].
\end{aligned} \tag{213}
$$

If we compare this with the action of the symplectically embedded Proca field we immediately recognize the similarities between the two theories. The main difference is that where the Proca theory had a parameter (m), now we have a field $(\eta + a)$. We wish to see how the Hamiltonian looks like in terms of the new variables. To do this, first we do the 3+1 decomposition of the above Lagrangian. Repeating the steps of the previous section first we define the canonical momenta

$$\underline{\Pi}^a = \frac{\delta L}{\delta \mathcal{L}_t \underline{A}_a} = \underline{E}^a \tag{214}$$

$$\pi_\eta = \frac{\delta L}{\delta \mathcal{L}_t \eta} = \sqrt{q}\frac{\mathcal{L}_t \eta - N^a \partial_a \eta}{N} \tag{215}$$

$$\pi_\Theta = \frac{\delta L}{\delta \mathcal{L}_t \Theta} = \left(\frac{a+\eta}{a}\right)^2 \sqrt{q}\left(\frac{\mathcal{L}_t\Theta - N^a\partial_a\Theta + aeA_0 - aeN^a\underline{A}_a}{N}\right) \tag{216}$$

and after the Legendre-transformation we obtain the Hamiltonian

$$H_{mat} = \int d^3x (N\mathcal{H}_{mat} + N^a\mathcal{H}_a^{mat} + A_0\underline{G}) \tag{217}$$

$$\mathcal{H}_{mat} = q_{ab}\left(\frac{\underline{E}^a\underline{E}^b + \underline{B}^a\underline{B}^b}{2\sqrt{q}}\right) + \frac{\pi_\eta^2}{2\sqrt{q}} + \frac{1}{2}\sqrt{q}\partial_a\eta\partial_a\eta + $$
$$+ \left(\frac{a}{a+\eta}\right)^2\frac{\pi_\Theta^2}{2\sqrt{q}} + \frac{1}{2}\sqrt{q}\left(\frac{a+\eta}{a}\right)^2(\partial_a\Theta + ae\underline{A}_a)^2 + \frac{1}{4}\sqrt{q}\mu\eta^2(2a+\eta)^2 \tag{218}$$

$$\mathcal{H}_a^{mat} = \epsilon_{abc}\underline{B}^c\underline{E}^b + \pi_\eta\partial_a\eta + \pi_\Theta\partial_a\Theta + ae\underline{A}_a\pi_\Theta \tag{219}$$

$$\underline{G} = D_a\underline{E}_a - ae\pi_\Theta \tag{220}$$

The (non-smeared, non-trivial) Poisson-brackets:

$$\{\pi_\eta, \eta\} = \frac{1}{2}\delta(x,y) \tag{221}$$

$$\{\pi_\Theta, \Theta\} = \frac{1}{2}\delta(x,y) \tag{222}$$

$$\{\underline{E}_a, \underline{A}^b\} = \delta_a^b\delta(x,y) \tag{223}$$

Now if one compares the Hamiltonian (217) with the original (205), it is easy to see that the two are connected with the help of the following canonical transformation:

$$\Phi := (a+\eta)\exp\left(\frac{i\Theta}{a}\right)$$
$$\Phi^* := (a+\eta)\exp\left(-\frac{i\Theta}{a}\right)$$
$$\pi := \left(\pi_\eta - \frac{ia}{a+\eta}\pi_\Theta\right)\exp\left(-\frac{i\Theta}{a}\right)$$

$$\pi^* \quad := \quad \left(\pi_\eta + \frac{ia}{a+\eta}\pi_\Theta\right)\exp\left(\frac{i\Theta}{a}\right)$$

Further more the above system is very similar to the case of the symplectically embedded Proca-field. To see this, let us introduce the canonical transformation $\pi_\Theta \to \frac{\pi_\Theta}{ea}$, $\Theta \to ea\Theta$ and define $m^2 = e^2(a+\eta)^2$. Then we will obtain exactly the Hamiltonian of Proca field, with the exception of a potential term. There are two major differences: there is an extra dynamical scalar field in the theory and the "mass" is constructed from the field η. The later will be quite important since after quantization all the fields will become operators so we can define a "mass operator", which spectrum can be identified as the mass spectrum (in the case of the Proca-field the mass was a parameter of the theory).

6.2.3 Classical symmetry breaking

In quantum field theory we use the unitary gauge to do gauge fixing. In the U(1) case this means we introduce the gauge-fixed vector field

$$\underline{\tilde{A}}_a^{(4)}(x) := \underline{A}_a^{(4)}(x) - \frac{1}{ea}\partial_a^{(4)}\Theta(x). \tag{224}$$

Substituting this into the Lagrangian we get

$$\tilde{L}_{mat} = \int d^4x\sqrt{-g}\left[-\frac{1}{4}\underline{\tilde{F}}_{ab}^{(4)}\underline{\tilde{F}}^{(4)ab} - \frac{1}{2}\partial_a^{(4)}\eta\partial^{(4)a}\eta - \frac{1}{2}e^2(a+\eta)^2\underline{\tilde{A}}_a^{(4)}\underline{\tilde{A}}^{(4)a} - \right.$$
$$\left. -\frac{1}{4}\mu\eta^2(2a+\eta)^2\right] \tag{225}$$

Again we want to see how the Hamiltonian changes, so we do the 3+1 decomposition as we did in the previous sections. The canonical momenta will be

$$\begin{aligned}
\underline{\tilde{\Pi}}^a &= \frac{\delta L}{\delta\mathcal{L}_t\underline{\tilde{A}}_a} = \underline{\tilde{E}}^a \\
\pi_\eta &= \frac{\delta L}{\delta\mathcal{L}_t\eta} = \sqrt{q}\frac{\mathcal{L}_t\eta - N^a\partial_a\eta}{N},
\end{aligned}$$

and the constraints will be

$$\mathcal{H}_{mat} \quad = \quad q_{ab}\frac{\underline{\tilde{E}}^a\underline{\tilde{E}}^b + \underline{\tilde{B}}^a\underline{\tilde{B}}^b}{2\sqrt{q}} + \frac{\pi_r^2}{2\sqrt{q}} + \frac{1}{2}\sqrt{q}\partial_a\eta\partial_a\eta +$$

$$
\begin{aligned}
&+ \quad \frac{1}{4}\sqrt{q}\mu\eta^2(2a+\eta)^2 + \frac{1}{2}e^2\sqrt{q}(a+\eta)^2(\underline{\tilde{A}}_a\underline{\tilde{A}}^a + (\frac{\tilde{A}_0 - N^a\underline{\tilde{A}}_a}{N})^2) && (226)\\
H_a^{mat} &= \epsilon_{abc}\underline{\tilde{B}}^c\underline{\tilde{E}}^b + \pi_r\partial_a\eta + e^2\underline{\tilde{A}}_a\sqrt{q}(a+\eta)^2\frac{\tilde{A}_0 - N^a\underline{\tilde{A}}_a}{N} && (227)\\
\underline{G} &= D_a\underline{\tilde{E}}_a - e^2\sqrt{q}(a+\eta)^2\frac{\tilde{A}_0 - N^a\underline{\tilde{A}}_a}{N} && (228)
\end{aligned}
$$

The (non-smeared, non-trivial) Poisson-brackets remain the same:

$$
\begin{aligned}
\{\pi_r, \eta\} &= \frac{1}{2}\delta(x,y) && (229)\\
\{\tilde{E}_a, \tilde{A}^b\} &= \delta_a^b\delta(x,y) && (230)
\end{aligned}
$$

If we compare this gauge fixed Hamiltonian with (217), we can see that in the Hamiltonian formalism the gauge fixing is equivalent to the introduction of the following two constraints

$$
\begin{aligned}
C_a &:= \partial_a Arg(\Phi) = 0 && (231)\\
C &:= a\pi_f - e\sqrt{q}(a+\eta)^2\frac{\tilde{A}_0 - N^a\tilde{A}_a}{N} = 0, && (232)
\end{aligned}
$$

(the second is equivalent to $\mathcal{L}_t\Theta = 0$). This is precisely the gauge we used in the case of the symplectically embedded Proca field. There we showed that in LQG gauge fixing is not necessary, in fact it makes the quantization extremely difficult if not impossible. So we will not fix the gauge, instead we will try to solve the constraint related to it.

To conclude we summarize the most important results of this section.

We checked how one can implement spontaneous symmetry breaking in the Hamiltonian formalism. It turned out that introducing new variables means a canonical transformation, while gauge fixing (as it was shown earlier e.g. in [48]) can be done by introducing new constraints. Interestingly these are exactly the same conditions which were introduced in the case of the Proca field. Further more the Hamiltonian (217) is very similar to the symplectically embedded Proca Hamiltonian, the only two exception is that we have an extra scalar field and the mass is not a parameter but defined with the help of the field η.

6.3 Quantization

The quantization of the gravitational and Maxwell fields are completely the same as in the case of the Proca-field, but the method for the scalar field needs to be modified. The crucial point of quantizing the scalar field is (see [34] or [46],[47]) that the field should be real valued. In our case the original variables are complex, but this does not cause significant difficulties since we can introduce new fields which are real, thus the usual techniques can be applied on them. The only non-trivial problem is an additional ambiguity which arises because this can be done more than one way. What we are going to do is introduce two kinds of different choices for the configuration variables and the momentum operators.

Case A:

The most natural choice is to define the operators with the help of the real and imaginary parts of the fields. Let v be a vertex of a graph γ with coordinates x_v. Then let

$$U(\lambda, v) \;:=\; \exp(i\lambda\Re(\Phi(x_v))) \tag{233}$$

$$\bar{U}(\delta, v) \;:=\; \exp(i\delta\Im(\Phi(x_v))), \tag{234}$$

where λ and δ are arbitrary real numbers which are required because otherwise the quantization would not be general enough (see [46] and [47] for details). The variables for the momentum operator should be (B is an open ball in Σ)

$$\Pi(B) = \int_B d^3x \Re(\pi)$$

$$\bar{\Pi}(B) = \int_B d^3x \Im(\pi)$$

and thus the Poisson-brackets of the variables will be

$$\{\Pi(B), U(\lambda, v)\} = \delta_{v\cap B,v}\frac{i\lambda}{2}U(\lambda, v) \tag{235}$$

$$\{\bar{\Pi}(B), \bar{U}(\delta, v)\} = -\delta_{v\cap B,v}\frac{i\delta}{2}\bar{U}(\delta, v) \tag{236}$$

$$\{\Pi(B), \bar{U}(\delta, v)\} = \{\bar{\Pi}(B), U(\lambda, v)\} = 0 \tag{237}$$

The transformation rules of these quantities with respect to the (smeared) gauge transformation ($\underline{G}(\Lambda) = \int d^3x \underline{G}\Lambda$) are:

$$
\begin{aligned}
\{\underline{G}(\Lambda), U(\lambda, v)\} &= ie\Lambda\lambda\Im(\Phi(x_v))U(\lambda, v) \\
\{\underline{G}(\Lambda), \bar{U}(\delta, v)\} &= -ie\Lambda\delta\Re(\Phi(x_v))\bar{U}(\delta, v)
\end{aligned}
$$

Since we have two fields, the Hilbert space for the scalar field is a tensor product $\mathcal{H}_{sc} = \mathcal{H}(U) \bigotimes \mathcal{H}(\bar{U})$, where the Hilbert spaces $\mathcal{H}(U)$ and $\mathcal{H}(\bar{U})$ are the linear combination of the following monomials: let $\underline{v} = v_1, \ldots, v_N$ be the set of vertices for some γ graph and let $\underline{\lambda}$ and $\underline{\bar{\delta}}$ be two sets of real numbers, each pair associated to a vertex. Then a basic element of $\mathcal{H}(U)$ is constructed as follows:

$$
|\underline{\lambda}\rangle_\gamma = \prod_{k=1}^{N} U(\lambda_k, v_k) \tag{238}
$$

In a similar fashion

$$
|\underline{\bar{\delta}}\rangle_\gamma = \prod_{k=1}^{N} \bar{U}(\bar{\delta}_k, v_k) \tag{239}
$$

will be a basic element is $\mathcal{H}(\bar{U})$. Both $|\underline{\lambda}\rangle_\gamma$ and $|\underline{\bar{\delta}}\rangle_\gamma$ form a complete orthonormal basis, that is ${}_{\gamma'}\langle\underline{\lambda}'|\underline{\lambda}\rangle_\gamma = \delta_{\underline{\lambda},\underline{\lambda}'}\delta_{\gamma,\gamma'}$ and the same is true for $|\underline{\bar{\delta}}\rangle_\gamma$.

Thus elements of $\mathcal{H}_{sc}$ are linear combinations of of monomials $|\underline{\lambda}\rangle_\gamma|\underline{\bar{\delta}}\rangle_\gamma$.

The operators are defined in the same way as in the case of gauge fields:

$$
\begin{aligned}
\hat{U}(\lambda, v)|\underline{\lambda}\rangle_\gamma &:= U(\lambda, v)|\underline{\lambda}\rangle_\gamma & (240) \\
\hat{\Pi}(B)|\underline{\lambda}\rangle_\gamma &:= i\bar{h}\{\Pi(B), |\underline{\lambda}\rangle_\gamma\} & (241) \\
\hat{\bar{U}}(\delta, v)|\underline{\bar{\delta}}\rangle_\gamma &:= \bar{U}(\delta, v)|\underline{\bar{\delta}}\rangle_\gamma & (242) \\
\hat{\bar{\Pi}}(B)|\underline{\bar{\delta}}\rangle_\gamma &:= i\bar{h}\{\bar{\Pi}(B), |\underline{\bar{\delta}}\rangle_\gamma\} & (243)
\end{aligned}
$$

Because the U(1) group is commutative, the action of the operators are very simple:

$$
\begin{aligned}
\hat{\Pi}(B)|\underline{\lambda}\rangle_\gamma &= -\frac{\bar{h}}{2}\sum_{v_j \in B} \lambda_j |\underline{\lambda}\rangle_\gamma & (244) \\
\hat{U}(\lambda, v)|\underline{\lambda}\rangle_\gamma &= |\underline{\lambda}'\rangle_\gamma
\end{aligned}
$$

$$\lambda_i' = \lambda_i + \delta_{v,v_i}\lambda \tag{245}$$

and similar expressions hold for $\hat{\underline{\hat{U}}}(\delta,v)$ and $\hat{\underline{\hat{\Pi}}}(B)$. Also, because of 237 $\hat{\underline{\hat{\Pi}}}(B)|\underline{\lambda}\rangle_\gamma = \hat{\Pi}(B)|\underline{\bar{\delta}}\rangle_\gamma = 0$.

Case B:

Another way is to use the absolute value and the argument of Φ. Actually these are equal (up to constant factors) with the fields η and Θ respectively, so we suggest the following operators for the multiplication operators:

$$U_\eta(\lambda,v) := \exp(i\lambda\eta(x_v)) \tag{246}$$

$$U_\Theta(\delta,v) := \exp(i\delta\frac{\Theta(x_v)}{a}). \tag{247}$$

For the momentum operators it is plausible to use the quantities π_η and π_Θ instead of $\Re(\Pi)$ and $\Im(\Pi)$:

$$\Pi_\eta(B) = \int_B d^3x\pi_\eta$$

$$\Pi_\Theta(B) = a\int_B d^3x\pi_\Theta$$

The Poisson-brackets of these variables are a bit different then in case A:

$$\{\Pi_\eta(B), U_\eta(\lambda,v)\} = i\frac{1}{2}\lambda\delta_{v\cap B,v}U_\eta(\lambda,v) \tag{248}$$

$$\{\Pi_\Theta(B), U_\Theta(\delta,v)\} = i\frac{1}{2}\delta\delta_{v\cap B,v}U_\Theta(\delta,v) \tag{249}$$

$$\{\Pi_\eta(B), U_\Theta(\delta,v)\} = \{\Pi_\Theta(B), U_\eta(\lambda,v)\} = 0 \tag{250}$$

The transformation rule for these variables with respect to gauge transformations are:

$$\{\underline{G}(\Lambda), U_\eta(\lambda,v)\} = 0$$

$$\{\underline{G}(\Lambda), U_\Theta(\delta,v)\} = -\frac{1}{2}ie\Lambda\delta\Theta(x_v)U_\Theta(\delta,v),$$

which means that $U_\eta(\lambda, v)$ is gauge invariant and the transformation rule for $U_\Theta(\delta, v)$ is

$$U_\Theta(\delta, v) \mapsto U_\Theta(\delta, v) U_\Theta(\frac{ae\Lambda\delta}{2}, v)^{-1}. \tag{251}$$

The construction of the phase space is completely identical to the construction in case A, the only difference is that one has to replace the old variables with the new ones. To avoid confusion, $\mathcal{H}_{sc}^{new} = \mathcal{H}(U_\eta) \bigotimes \mathcal{H}(U_\Theta)$ will stand for the new phase space,

$$|\underline{\lambda}^\eta\rangle_\gamma = \prod_{k=1}^{N} U_\eta(\lambda_k^\eta, v_k) \tag{252}$$

will label an element of $\mathcal{H}(U_\eta)$ and

$$|\underline{\delta}^\Theta\rangle_\gamma = \prod_{k=1}^{N} U_\Theta(\delta_k^\Theta, v_k) \tag{253}$$

will be an element of $\mathcal{H}(U_\Theta)$. The action of these operators are completely the same as in case A:

$$\hat{\Pi}_\eta(B)|\underline{\lambda}^\eta\rangle_\gamma = -\frac{\hbar}{2} \sum_{v_j \in B} \lambda_j^\eta |\underline{\lambda}^\eta\rangle_\gamma \tag{254}$$

$$\begin{aligned} \hat{U}_\eta(\lambda, v)|\underline{\lambda}^\eta\rangle_\gamma &= |\underline{\lambda}'^\eta\rangle_\gamma \\ \lambda_i'^\eta &= \lambda_i^\eta + \delta_{v,v_i}\lambda \end{aligned} \tag{255}$$

6.3.1 Regularisation

In order to quantize this system, one first has to rewrite the Hamiltonian in terms of the variables defined in the previous section. Fortunately the terms we have are very similar to the terms derived already in section 4 so we need to concentrate only on those terms that are different. Specifically these are the terms that contain the scalar field. We will deal with the two kinds of description (the original case with Φ and Φ^* and the case with new variables η and Θ) separately.

Case A:

Although later we will use the formulas involving η and Θ we shall provide the regularisation of the original Hamiltonian, since it has some non trivial steps. The potential term is the simplest: since $\Phi\Phi^* = \Re(\Phi)^2 + \Im(\Phi)^2$ and $\Re(\Phi) = \arccos\left(\frac{U(\lambda,v)+U^{-1}(\lambda,v)}{2}\right)$, we can write this (using the notations introduced earlier) in the following form:

$$\hat{H}_{pot} = \frac{1}{4}\sum_v N(v)\mu\hat{V} \times$$
$$\times \left[\frac{1}{\lambda^2}\arccos\left(\frac{U(\lambda,v)+U^{-1}(\lambda,v)}{2}\right)^2 + \frac{1}{\delta^2}\arccos\left(\frac{\bar{U}(\delta,v)+\bar{U}^{-1}(\delta,v)}{2}\right)^2 - a^2\right]^2 \tag{256}$$

One may wonder why we used the arccos function instead of e.g. the logarithm. The main reason is that since spontaneous symmetry breaking requires the ground state of the potential, we are *forced* to regularize the potential term to be self-adjoint. It is easy to see that the above operator is self-adjoint, but this would not be the case if we used the logarithm function. Of course there are still ambiguities in the regularisation, but this certainly narrows down the possibilities.

In a similar fashion, one replaces $\pi\pi^* = \Re(\pi)^2 + \Im(\pi)^2$ in the kinetic term to obtain

$$\hat{H}_P = \frac{1}{2}\sum_v N(v)\frac{X(v)^2 + \bar{X}(v)^2}{E(v)^2}\hat{G}_1(v), \tag{257}$$

where $X(v)$ and $\bar{X}(v)$ are the invariant vector fields on U(1) and $\hat{G}_1(v)$ contains only gravitational variables and it is the same as in section 4:

$$\begin{aligned}
\hat{G}_1(v) &= \frac{8}{81m^2\bar{h}^4\kappa^6}\sum_{v(\Delta)=v(\Delta`)=v}\epsilon^{IJK}\epsilon^{LMN}\epsilon_{ijk}\epsilon_{lmn} \times \\
&\times \hat{Q}^i_{s_I(\Delta)}(v,\frac{1}{2})\hat{Q}^j_{s_J(\Delta)}(v,\frac{1}{2})\hat{Q}^k_{s_K(\Delta)}(v,\frac{1}{2}) \times \\
&\times \hat{Q}^l_{s_L(\Delta`)}(v,\frac{1}{2})\hat{Q}^m_{s_M(\Delta`)}(v,\frac{1}{2})\hat{Q}^n_{s_N(\Delta`)}(v,\frac{1}{2}),
\end{aligned} \tag{258}$$

where $\hat{Q}^k_e(v,r) = tr(\tau_k h_e[h_e^{-1},\hat{V}(v)^r])$, h_e being the holonomy of the Ashtekar connection along edge e, $\hat{V}$ is the volume operator and τ_k are the generators of SU(2).

The derivative term needs a more careful treatment. First we have to rewrite it in terms of $\Re(\Phi)$ and $\Im(\Phi)$

$$D_a\Phi(D_b\Phi)^* = (\partial_a + ieA_a)(\Re(\Phi) + i\Im(\Phi))(\partial_b - ieA_b)(\Re(\Phi) - i\Im(\Phi))$$

From this we can see that we need to regularize the expression $(\partial_a \pm ieA_a)\Re(\Phi)$. This is quite similar to the derivative term $\partial_a\Phi \pm A_a$, the only difference is that we have a $iA_a\Re(\Phi)$ term instead of A_a. Though this seems a minor change, it turns out that the regularized expression for this covariant derivative is more complicated, which is due to the fact that it contains the multiplication of the two fields. We can overcome this difficulty by doing the regularisation in a step-by-step way. First we note that for small Δt

$$h_s = 1 + ie\Delta t\dot{s}^a A_a + o(\Delta t^2)$$

for an edge s. This means that (v is the beginning of the edge s)

$$(h_s - 1)\arccos\Big(\frac{U(\lambda, v) + U(\lambda, v)^{-1}}{2}\Big) = ie\lambda\Delta t\dot{s}^a A_a\Re(\Phi) + o(\Delta t^2),$$

so if we take into account that

$$U(\lambda, s(\Delta t)) = 1 + i\lambda(\Re(\Phi) + \Delta t\dot{s}^a\partial_a\Re(\Phi)) + o(\Delta t^2),$$

we arrive to the following regularized expression:

$$\begin{aligned}
& U(\lambda, s(\Delta t))[1 + i(h_s - 1)\arccos\Big(\frac{U(\lambda, v) + U(\lambda, v)^{-1}}{2}\Big)]U(\lambda, v)^{-1} = \\
= \; & [1 + i\lambda(\Re(\Phi) + \Delta t\dot{s}^a\partial_a\Re(\Phi))](1 - e\lambda\Delta t\dot{s}^a A_a\Re(\Phi))[1 - i\lambda\Re(\Phi)] + o(\Delta t^2) = \\
= \; & 1 + i\lambda\Delta t\dot{s}^a(\partial_a\Re(\Phi)) + ieA_a\Re(\Phi))) + o(\Delta t^2).
\end{aligned} \tag{259}$$

We obtain the same result for $(\partial_a + ieA_a)\Im(\Phi)$ if we replace U with $\bar{U}$. Also the term $(\partial_a - ieA_a)\Re(\Phi)$ is obtained by replacing h_s with h_s^{-1}. To simplify the result let us introduce a notation:

$$\begin{aligned}
W(v, s, \lambda) &= \frac{1}{\lambda}U(\lambda, s(\Delta t))[1 + i(h_s - 1)\arccos\Big(\frac{U(\lambda, v) + U(\lambda, v)^{-1}}{2}\Big)]U(\lambda, v)^{-1} - \frac{1}{\lambda} \\
\bar{W}(v, s, \delta) &= \frac{1}{\delta}\bar{U}(\delta, s(\Delta t))[1 + i(h_s - 1)\arccos\Big(\frac{\bar{U}(\delta, v) + \bar{U}(\delta, v)^{-1}}{2}\Big)]\bar{U}(\delta, v)^{-1} - \frac{1}{\delta}.
\end{aligned}$$

Using the fact that $h_s^{-1} = h_{s^{-1}}$, the regulated expressions are the following:

$$W(v, s, \lambda) = i\Delta t\dot{s}^a(\partial_a\Re(\Phi)) + ieA_a\Re(\Phi))) + o(\Delta t^2)$$

$$
\begin{aligned}
W(v,s^{-1},\lambda) &= i\Delta t\dot{s}^a(\partial_a\Re(\Phi)) - ieA_a\Re(\Phi))) + o(\Delta t^2)\\
\bar{W}(v,s,\delta) &= i\Delta t\dot{s}^a(\partial_a\Im(\Phi)) + ieA_a\Im(\Phi))) + o(\Delta t^2)\\
\bar{W}(v,s^{-1},\delta) &= i\Delta t\dot{s}^a(\partial_a\Im(\Phi)) - ieA_a\Im(\Phi))) + o(\Delta t^2)
\end{aligned}
$$

This way the derivative term will be the limit of

$$
\begin{aligned}
&\hat{H}_{der} = \frac{1}{2}\sum_v \frac{N(v)}{E(v)^2}\times\\
&\times \sum_{v(\Delta)=v(\Delta`)=v}[W_\Delta(v,s_n,\lambda)+i\bar{W}_\Delta(v,s_n,\delta)][W_{\Delta'}(v,s_r^{-1},\lambda)-i\bar{W}_{\Delta'}(v,s_r^{-1},\delta)]\hat{G}_2^{nr}(v),
\end{aligned}
$$

where

$$
\hat{G}_2^{nr}(v) = \frac{1}{2\bar{h}^4\kappa^4}\left(\frac{4}{3}\right)^6 \epsilon^{ijk}\epsilon^{ilm}\epsilon_{npq}\epsilon_{rst}\hat{Q}^j_{s_p(\Delta)}(v,\frac{3}{4})\hat{Q}^k_{s_q(\Delta)}(v,\frac{3}{4})\hat{Q}^l_{s_s(\Delta`)}(v,\frac{3}{4})\hat{Q}^m_{s_t(\Delta`)}(v,\frac{3}{4})
$$

and the Δ and Δ' subscripts represent the tetrahedra where the holonomies and pointholonomies should be calculated.

Case B: In this case one should be careful since the two scalar fields do not appear in a symmetric way. For instance, the potential term contains only η, so one simply replaces $\eta = \frac{1}{\lambda}\arccos\left(\frac{U_\eta(\lambda,v)+U_\eta^{-1}(\lambda,v)}{2}\right)$ to obtain

$$
\begin{aligned}
\hat{H}_{pot} &= \frac{1}{4}\sum_v N(v)\mu\hat{V}\frac{1}{\lambda^2}\arccos\left(\frac{U_\eta(\lambda,v)+U_\eta^{-1}(\lambda,v)}{2}\right)^2\times\\
&\times \left[\frac{1}{\lambda}\arccos\left(\frac{U_\eta(\lambda,v)+U_\eta^{-1}(\lambda,v)}{2}\right)+2a\right]^2
\end{aligned}
\tag{260}
$$

The terms containing π_η and π_Θ can be treated in the same way as in the previous case, one just has to be careful since the later contains the expression $\frac{1}{(a+\eta)^2}$. But it is easy to see that if one carries out the regularisation procedure as earlier the only difference will be a term which is the above fraction expressed with the variables U_η. Thus the result for the two kinetic terms will be

$$
\hat{H}_P = \frac{1}{2}\sum_v N(v)\frac{X_\eta(v)^2+\left[\frac{1}{\lambda}\arccos\left(\frac{U_\eta(\lambda,v)+U_\eta^{-1}(\lambda,v)}{2}\right)+a\right]^{-2}X_\Theta(v)^2}{E(v)^2}\hat{G}_1(v). \tag{261}
$$

The derivative terms for the two scalar fields are also different. In the case of η, one only needs the expression (for small Δt):

$$U_\eta(\lambda, s(\Delta t))U_\eta^{-1}(\lambda, v) - 1 = i\lambda\Delta t\dot{s}^a\partial_a\eta + o(\Delta t^2), \tag{262}$$

thus it has contribution to the Hamiltonian

$$\begin{aligned}\hat{H}_{der}(\eta) &= \frac{1}{2}\sum_v \frac{N(v)}{E(v)^2}\sum_{v(\Delta)=v(\Delta`)=v}\times \\ &\times \frac{1}{\lambda^2}[U_\eta(\lambda, s_n(\Delta))U_\eta^{-1}(\lambda, v) - 1][U_\eta(\lambda, s_r(\Delta'))U_\eta^{-1}(\lambda, v) - 1]\hat{G}_2^{mr}(v) \end{aligned} \tag{263}$$

For the field Θ we remind the reader that in the case of the Proca-field the same kind of coupling appeared between the scalar field and the Maxwell field. Thus we may use the approximation mentioned there:

$$U_\Theta(\delta, s(\Delta t))\underline{h}_s U_\Theta(\delta, v)^{-1} - 1 = i\delta\Delta t\dot{s}^b(\frac{\partial_b\Theta}{a} + e\underline{A}_b) \tag{264}$$

Treating the term $(a+\eta)^2$ as before, the regulated expression of this term will be

$$\begin{aligned}\hat{H}_{der}(\Theta) &= \frac{1}{2}\sum_v \frac{N(v)}{E(v)^2}\Big[\frac{1}{\lambda}\arccos\Big(\frac{U_\eta(\lambda, v) + U_\eta^{-1}(\lambda, v)}{2}\Big) + a\Big]^2\sum_{v(\Delta)=v(\Delta`)=v}\times \\ &\times \frac{1}{\delta^2}[U_\Theta(\delta, s_n(\Delta))\underline{h}_{s_n}U_\Theta(\delta, v)^{-1} - 1][U_\Theta(\delta, s_r(\Delta'))\underline{h}_{s_r}U_\Theta(\delta, v)^{-1} - 1]\hat{G}_2^{mr}(v) \end{aligned} \tag{265}$$

6.4 New basis

In contrast to the Proca field, the mass here is represented by an operator, namely

$$\hat{m}(v) = e\Big[\frac{1}{\lambda}\arccos\Big(\frac{U_\eta(\lambda, v) + U_\eta^{-1}(\lambda, v)}{2}\Big) + a\Big]. \tag{266}$$

Since we want to compare the two theories, it would be useful to work in a basis where $U_\eta(\lambda, v)$ is diagonal and this is what we are going to do in this section.

6.4.1 The spectrum of $U_\eta(\lambda, v)$

Let $|\phi\rangle := |\lambda^\eta_{v_1}, \ldots, \lambda^\eta_{v_N}\rangle$ be a base element for a γ graph which has N vertices The action of $U_\eta(\lambda, v)$ on this state is

$$U_\eta(\lambda, v)|\phi\rangle = |\lambda^\eta_{v_1}, \ldots, \lambda^\eta_{v_k} + \lambda, \ldots, \lambda^\eta_{v_N}\rangle\delta(v, v_k) \tag{267}$$

This action suggests that we should look for eigenstates in the form

$$|\Lambda^\eta(\lambda, v), \underline{\lambda}^\eta\rangle := \sum_{i=-\infty}^{\infty} \Big(\frac{U_\eta(\lambda, v)}{\Lambda^\eta(\lambda, v)}\Big)^i |\underline{\lambda}^\eta\rangle, \tag{268}$$

where $|\underline{\lambda}^\eta\rangle$ is an arbitrary state and $\Lambda^\eta(\lambda, v)$ is a (yet) arbitrary number (this will be the eigenvalue for a given λ at a vertex v). It is easy to verify that

$$U_\eta(\lambda, v)|\Lambda^\eta(\lambda, v), \underline{\lambda}^\eta\rangle = \Lambda(\lambda, v)|\Lambda^\eta(\lambda, v), \underline{\lambda}^\eta\rangle. \tag{269}$$

We shall call these one vertex eigenstates because $|\Lambda^\eta(\lambda, v), \underline{\lambda}^\eta\rangle$ is the eigenstate of only those $U_\eta(\lambda, v')$ where v'=v. Since $U_\eta(\lambda, v)$ is unitary, we can write $\Lambda^\eta(\lambda, v)$ in the following form: $\Lambda^\eta(\lambda, v) = \exp(i\Delta^\eta(\lambda, v))$, where Δ^η is real.In fact, since $U_\eta(0, v) = \hat{1}$ and $U_\eta(\lambda_1, v)U_\eta(\lambda_2, v) = U_\eta(\lambda_1 + \lambda_2, v)$, we obtain that $\Delta^\eta(\lambda, v)$ is of the form $\Delta^\eta(\lambda, v) = \Gamma^\eta(v)\lambda$. In summary, the spectrum of $U_\eta(\lambda, v)$ is of the form $\exp(i\lambda\Gamma^\eta(v))$, so instead of $\Lambda^\eta(\lambda, v)$ we shall use $\Gamma^\eta(v)$. We can select an orthonormal basis from these eigenstates in the following way. Note that if there exists an integer n such that $|\underline{\lambda}^\eta_1\rangle = U(\lambda, v)^n|\underline{\lambda}^\eta_2\rangle$ then $|\Gamma^\eta(v), \underline{\lambda}^\eta_1\rangle = e^{in\lambda\Gamma^\eta(v)}|\Gamma^\eta(v), \underline{\lambda}^\eta_2\rangle$. Because of this let us restrict ourselves to those $|\underline{\lambda}^\eta\rangle$ that satisfy the condition $0 \leq \lambda^\eta_v\langle\lambda$. Further more if we restrict the values of $\Gamma^\eta(v)$ so that $0 \leq \Gamma^\eta(v)\langle\frac{2\pi}{\lambda}$, these states will form a complete orthonormal basis in the sense

$$\begin{aligned}
&\langle\Gamma^\eta_1(v), \underline{\lambda}_1|\Gamma^\eta_2(v), \underline{\lambda}_2\rangle = \\
= \;& \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} \langle\lambda^{\eta,1}_{v_1}, \ldots, \lambda^{\eta,1}_v + k\lambda, \ldots, \lambda^{\eta,1}_{v_N}|\lambda^{\eta,2}_{v_1}, \ldots, \lambda^{\eta,2}_v + j\lambda, \ldots, \lambda^{\eta,2}_{v_M}\rangle \times \\
\times \;& \exp(i\lambda(j\Gamma^\eta_2(v) - k\Gamma^\eta_1(v))) = \delta(\underline{\lambda}^\eta_1, \underline{\lambda}^\eta_2) \sum_{k=-\infty}^{\infty} \exp(i\lambda k(\Gamma^\eta_2(v) - \Gamma^\eta_1(v))) = \\
= \;& \delta(\underline{\lambda}^\eta_1, \underline{\lambda}^\eta_2)\delta(\Gamma^\eta_2(v) - \Gamma^\eta_1(v))),
\end{aligned} \tag{270}$$

where $\delta(\underline{\lambda}_1^\eta, \underline{\lambda}_2^\eta) = \delta(\lambda_1^{(1)\eta} - \lambda_2^{(1)\eta}) \dots \delta(\lambda_1^{(N)\eta} - \lambda_2^{(N)\eta})$. To see that this is a *complete* orthonormal basis, one only has to check whether each original basis element can be expressed as the linear combination of the eigenstates. Let us suppose then that there exist complex numbers $C_{\underline{\lambda}}(\Gamma^\eta(v))$ such that

$$\sum_{\underline{\lambda}^\eta} \int d\Gamma^\eta(v) C_{\underline{\lambda}^\eta}(\Gamma^\eta(v)) |\Gamma^\eta(v), \underline{\lambda}^\eta\rangle = |\underline{\lambda}^{\eta,}\rangle \tag{271}$$

for each $|\underline{\lambda}^{\eta,}\rangle$. Because of orthogonality we obtain for the coefficients the following:

$$C_{\underline{\lambda}^\eta}(\Gamma^\eta(v)) = \langle \Gamma^\eta(v), \underline{\lambda}^\eta | \underline{\lambda}^{\eta,}\rangle = \langle \underline{\lambda}^\eta | \sum_{k=-\infty}^{\infty} \exp(-ik\lambda\Gamma^\eta(v)) U_\eta(\lambda, v)^k |\underline{\lambda}^{\eta,}\rangle \tag{272}$$

Now if for a $|\underline{\lambda}^\eta\rangle$ there exists an integer n such that $|\underline{\lambda}^\eta\rangle = U(\lambda, v)^n |\underline{\lambda}^{\eta,}\rangle$ then the corresponding coefficient will be

$$C_{\underline{\lambda}^\eta}(\Gamma^\eta(v)) = \exp(-in\lambda\Gamma^\eta(v)),$$

otherwise it is zero. It is easy to see that this correspondence is unique and since the original basis is complete, we verified our statement.

We define the graph eigenstate in a similar fashion. Let γ be a graph and $|\underline{\lambda}^\eta\rangle$ be an arbitrary state on that graph. For each vertex let $\Gamma^\eta(v_i)$ $i = 1 \dots N$ be a real number satisfying $0 \leq \Gamma^\eta(v_i) \langle \frac{2\pi}{\lambda}$. Then the graph eigenstate will be the following:

$$|\underline{\Gamma}^\eta, \underline{\lambda}^\eta\rangle := \sum_{k_1=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} (e^{ik_1\lambda\Gamma^\eta(v_1)} U_\eta(\lambda, v_1)^{k_1}) \dots (e^{ik_N\lambda\Gamma^\eta(v_N)} U_\eta(\lambda, v_N)^{k_N}) |\underline{\lambda}^\eta\rangle \tag{273}$$

Using the results obtained for the one vertex eigenstates we can find an orthonormal basis in the case of the graph eigenstates: if $0 \leq \lambda_v^\eta \langle \lambda$ for all v we get

$$\langle \underline{\Gamma}_1^\eta, \underline{\lambda}_1^\eta | \underline{\Gamma}_2^\eta, \underline{\lambda}_2^\eta\rangle = \delta(\underline{\lambda}_1^\eta, \underline{\lambda}_2^\eta) \prod_{k=1}^{N} \delta(\Gamma_1^\eta(v_1) - \Gamma_2^\eta(v_1)) \dots \delta(\Gamma_1^\eta(v_N) - \Gamma_2^\eta(v_N)) \tag{274}$$

Also we can express any state in terms of graph eigenstates with the help of the following expression:

$$\sum_{\underline{\lambda}^\eta} \int d\underline{\Gamma}^\eta C_{\underline{\lambda}^\eta}(\underline{\Gamma}^\eta)|\underline{\Gamma}^\eta, \underline{\lambda}^\eta\rangle = |\underline{\lambda}^{\eta,}\rangle \tag{275}$$

where

$$\int d\underline{\Gamma}^\eta = \int d\Gamma^\eta(v_1)\dots\int d\Gamma^\eta(v_N).$$

What remains is the action of the momentum operators on an eigenstate. This is easy because of the following:

$$\begin{aligned} X(v)|\Gamma^\eta(v'),\underline{\lambda}\rangle &= X(v)\sum_{k=-\infty}^{\infty}\Big(\frac{U_\eta(\lambda,v')}{\exp(i\lambda\Gamma^\eta(v'))}\Big)^k|\underline{\lambda}^\eta\rangle = \\ &= \delta_{v,v_k}\lambda_k|\Gamma^\eta(v'),\underline{\lambda}\rangle - i\delta_{v,v'}\sum_{k=-\infty}^{\infty} ik\lambda\Big(\frac{U_\eta(\lambda,v')}{\exp(i\lambda\Gamma^\eta(v'))}\Big)^k|\underline{\lambda}^\eta\rangle = \\ &= (\delta_{v,v_k}\lambda_k + i\delta_{v,v'}\frac{\delta}{\delta\Gamma(v')})|\Gamma^\eta(v'),\underline{\lambda}\rangle \end{aligned} \tag{276}$$

With a completely similar analysis one can show that

$$X(v)|\underline{\Gamma}^\eta,\underline{\lambda}\rangle = (\delta_{v,v_k}\lambda_k + i\delta_{v,v'}\frac{\delta}{\delta\Gamma(v')})|\underline{\Gamma}^\eta,\underline{\lambda}\rangle \tag{277}$$

6.5 Solution to the constraints

In section 4 we sketched how one could solve the constraints of the theory. In this section we will follow the same procedures mentioned there, but there will be one difference, namely that for the fields η and Θ we do not work in the usual basis, rather in the Fock-space. Since for the other fields the algorithm remains the same, we will concentrate only on the scalar fields. Solving the diffeomorphism- and gauge constraints will be rather simple, so we start with them. Then - in order to simplify things - we will introduce a compact notation where we separate the scalar fields from the others, which is described in Appendix B. This is motivated by the fact that the scalar constraint is quite complicated, but with the new notation the structure of the equation will be much easier to examine.

Let us start with the diffeomorphism constraint. As it was pointed out earlier the infinitesimal generator of the diffeomorphism constraint cannot be implemented in the quantum theory, thus the techniques used to solve the Gauss- or scalar constraint cannot be applied here. The strategy is to use group averaging to solve the constraint, which can be generalized to the case where matter fields also appear. Since these are applied only to graphs not to the labels means that it is independent whether we use the Fock-space or the dust network space.

The gravitational Gauss-constraint does not change, so we can solve it by restricting ourselves to gauge invariant spin network states.

The U(1) Gauss-constraint contains variables of the electromagnetic field and the scalar field Θ so we analyze it in detail. The (smeared) integrated constraint

$$\int_\sigma \underline{G}\Lambda = \int_\sigma \Lambda(D_a\underline{E}_a - ae\pi_\Theta) \tag{278}$$

can be regulated in the following way: if we look for solutions in the form

$$\Psi = \sum_{s,f,\underline{\lambda},\underline{\bar{\delta}}} \int d\underline{\Gamma} \int d\underline{\bar{\Gamma}} C_{s,f,\underline{\lambda},\underline{\bar{\delta}}}(\underline{\Gamma},\underline{\bar{\Gamma}})\langle \underline{s}|\langle \underline{f}|\langle \underline{\Gamma},\underline{\lambda}|\langle \underline{\bar{\Gamma}},\underline{\bar{\delta}}|. \tag{279}$$

then it can be verified that the quantum version of the above constraint is the following:

$$\langle\Psi|\sum_v \Lambda_v[\sum_{e\cap v=v} l_e - (\delta_v + i\frac{\delta}{\delta\bar{\Gamma}(v)})]|\Phi\rangle = 0 \tag{280}$$

for all spin color network state $|\Phi\rangle$. Here l_e is the integer on the edge e (this comes from the flux network). Since Λ_v is arbitrary the above equation is equivalent to

$$\int d\bar{\Gamma}(v)' C_{s,f,\underline{\lambda},\underline{\bar{\delta}}'}(\underline{\Gamma},\underline{\bar{\Gamma}}')\langle\underline{\bar{\Gamma}}',\underline{\bar{\delta}}'|\sum_{e\cap v=v} l_e - (\delta_v + i\frac{\delta}{\delta\bar{\Gamma}(v)})|\underline{\bar{\Gamma}},\underline{\bar{\delta}}\rangle = 0, \tag{281}$$

where we inserted (279) to the constraint equation and used orthogonality of spin color network states. Now after partial integration we obtain a (functional) differential equation on the coefficients $C_{s,f,\underline{\lambda},\underline{\bar{\delta}}}(\underline{\Gamma},\underline{\bar{\Gamma}})$:

$$[\sum_{e\cap v=v} l_e - (\delta_v - i\frac{\delta}{\delta\bar{\Gamma}(v)})]C_{s,f,\underline{\lambda},\underline{\bar{\delta}}}(\underline{\Gamma},\underline{\bar{\Gamma}}) = 0. \tag{282}$$

Since we have a similar equation for all v, the solution to this constraint is:

$$C_{s,f,\underline{\lambda},\underline{\bar{\delta}}}(\underline{\Gamma},\underline{\bar{\Gamma}}) = C_{s,f,\underline{\lambda},\underline{\bar{\delta}}}(\underline{\Gamma}) \prod_{v} \exp[-i(\sum_{e\cap v=v} l_e - \delta_v)\bar{\Gamma}(v)], \tag{283}$$

where the coefficients $C_{s,f,\underline{\lambda},\underline{\bar{\delta}}}(\underline{\Gamma})$ are arbitrary.
What remains is the scalar constraint. If we look at the Hamiltonian, it is clear that the constraint equation will be a differential equation with respect to the variable Γ. First we write down this equation. The condition we have to solve is

$$\langle\Psi|\hat{H}|\phi\rangle = 0 \tag{284}$$

for arbitrary $|\phi\rangle$. Again we can say that the support of N is at only one vertex v. Substituting (279) into the above equation we obtain

$$\begin{aligned}\sum_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'} \int d\Gamma(v)' \int d\bar{\Gamma}(v)' C_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}(\underline{\Gamma}',\underline{\bar{\Gamma}}') \times \\ \times \langle s'|\langle f'|\langle\underline{\Gamma}',\underline{\lambda}'|\langle\underline{\bar{\Gamma}}',\underline{\bar{\delta}}'|\hat{H}|\underline{\bar{\Gamma}},\underline{\bar{\delta}}\rangle|\underline{\Gamma},\underline{\lambda}\rangle|f\rangle|s\rangle = 0\end{aligned}$$

In the previous section we have shown a method (generalizing the results of [32]) which simplified the above equation by turning it into a finite number of equations and now we apply these results to the gravitational and electromagnetic fields, arriving to a finite system of linear differential equations. Since we concentrate only on the scalar field $|\Gamma,\lambda\rangle$ at the moment and the Hamiltonian contains several terms, we shall calculate each term separately and introduce a compact notation. This notation is introduced in appendix B where the reader will also find the terms of the scalar constraint. The conclusion is that the scalar constraint is actually a system of linear differential equations of second order:

$$\sum_{I'} H^P_{I'I}(\lambda_v - i\frac{\delta}{\delta\Gamma(v)})\tilde{C}_{I'}(\underline{\Gamma}) = -\sum_{I'} H_{I'I}(\underline{\Gamma})\tilde{C}_{I'}(\underline{\Gamma}) \tag{285}$$

where

$$\begin{aligned}&H_{I'I}(\underline{\Gamma}) = \\ =\ &\frac{L(v)^2 H^P_{I'I}(v)}{(\Gamma(v)+a)^2} + H^{G+YM}_{I'I}(v) + H^{der1}_{I'I}(\Gamma(v)+a)^2 + H^{pot}_{I'I}(v)\Gamma(v)^2(\Gamma(v)+2a)^2 +\end{aligned}$$

$$+ \quad H^A_{I'I}(v)\exp(-2i\lambda\Gamma(v)) - H^B_{I'I}(v)\exp(-i\lambda\Gamma(v)) + H^C_{I'I}(v) \tag{286}$$

To simplify this term we look for solutions of the form

$$C_I(\underline{\Gamma}) := \tilde{C}_I(\underline{\Gamma})\exp(-i\prod_v \lambda_v\Gamma(v)),$$

since

$$(\lambda_v - i\frac{\delta}{\delta\Gamma(v)})C_I(\underline{\Gamma}) = -i\Big(\frac{\delta}{\delta\Gamma(v)}\tilde{C}_I(\underline{\Gamma})\Big)\exp(-i\prod_v \lambda_v\Gamma(v)).$$

The other terms will also contain a factor $\exp(-i\prod_v \lambda_v\Gamma(v))$ so this drops out of the differential equation, leaving us with the following formula for $\tilde{C}_{I'}(\underline{\Gamma})$:

$$\sum_{I'} H^P_{I'I}\frac{\delta^2}{\delta\Gamma(v)^2}\tilde{C}_{I'}(\underline{\Gamma}) = \sum_{I'} H_{I'I}(\underline{\Gamma})\tilde{C}_{I'}(\underline{\Gamma}) \tag{287}$$

6.5.1 Solving the scalar constraint

This system of linear differential equations can be solved using the method we shown in Appendix C if the matrix $H^P_{I'I}$ is invertible. If it is not invertible then let us diagonalize the left hand side, i.e. find a unitary $\mathbf{U}$ such that $\mathbf{U}\mathbf{H}^P\mathbf{U}^{-1} = diag(k_1,\ldots,k_N)$ where $k_1,\ldots,k_N$ are the eigenvalues of $\mathbf{H}^P$. Let us order the eigenvalues in a way that $k_1,\ldots,k_M$ $(M\langle N)$ be all the zero eigenvalues. This means that the first M equation in this case is not a differential equation but only an algebraic equation. Since in this case the left hand side is zero, the right hand side is zero if and only $\sum_{I'=1}^M H'^Q_{I'I}(v)\tilde{C}_{I'} = 0$ etc. for all matrices appearing in $H_{I'I}(\underline{\Gamma})$ $(H'^Q_{I'I}(v) = (\mathbf{U}\mathbf{H}^Q(v)\mathbf{U}^{-1})_{I'I}$ etc.), which means that after solving the algebraic equations we again arrive to a system of linear differential equations but with an invertible matrix on the left hand side. So from now on we consider $H^P_{I'I}$ to be invertible.

To have a correct solution we must specify the initial condition on $\tilde{C}_{I'}(\underline{\Gamma})$ and $\frac{\partial}{\partial\Gamma(v)}\tilde{C}_{I'}(\underline{\Gamma})$. The fact that $\Gamma(v) = -a$ can be interpreted as the disappearance of the field η implies that

$$[\sum_{I'} H_{I'I}(\underline{\Gamma})\tilde{C}_{I'}(\underline{\Gamma})]_{|\Gamma(v)=-a} = 0 \tag{288}$$

be the first condition. With the same reasoning the second condition is that the momentum of the field should disappear. In this case (since all λ_v are zero) we arrive to the condition

$$[\frac{\partial}{\partial\Gamma(v)}\tilde{C}_{I'}(\underline{\Gamma})]_{|\Gamma(v)=-a} = 0 \tag{289}$$

Since $H_{I'I}$ has a complicated structure, the differential equation cannot be solved explicitly. However we can solve it in some special case.
First let us consider the case when $\Gamma(v) \approx -a$. In this case the system of differential equations takes the form

$$\sum_{I'} H^P_{I'I}\frac{\delta^2}{\delta\Gamma(v)^2}\tilde{C}_{I'}(\underline{\Gamma}) = \sum_{I'}\frac{L(v)^2 H^P_{I'I}(v)}{(\Gamma(v)+a)^2}\tilde{C}_{I'}(\underline{\Gamma}) + \sum_{I'} H^{G+YM}_{I'I}\tilde{C}_{I'}(-a) \tag{290}$$

Now if we multiply both sides with $(\mathbf{H}^P)^{-1}$ and define $b_I = ((\mathbf{H}^P)^{-1}\mathbf{H}^{G+YM}\vec{C}(-a))_I$ we get

$$\frac{\delta^2}{\delta\Gamma(v)^2}\tilde{C}_I(\underline{\Gamma}) - \frac{L(v)^2}{(\Gamma(v)+a)^2}\tilde{C}_I(\underline{\Gamma}) = b_I \tag{291}$$

The general solution of this differential equation is the following:

$$\tilde{C}_I(\underline{\Gamma}) = \frac{(\Gamma(v)+a)^2}{2-L(v)^2}b_I + C^1_I(\Gamma(v)+a)^{n_1} + C^2_I(\Gamma(v)+a)^{n_2}, \tag{292}$$

where

$$n^1_2 = \frac{1 \pm \sqrt{1+4L(v)^2}}{2}$$

and C^1_I, C^2_I are arbitrary constants. From $L(v)^2 \geq 0$ follows that $n_1 \geq 1$ and $n_2 \leq 0$, which means that if $L(v) \neq 0$ then $\tilde{C}_I(\underline{\Gamma})$ is singular in $\Gamma(v) = -a$.
Now let us consider the initial conditions. If $L(v) = 0$ then

$$\tilde{C}_I(\underline{\Gamma}) = \frac{(\Gamma(v)+a)^2}{2}b_I + C^1_I(\Gamma(v)+a) + C^2_I. \tag{293}$$

Substituting into (288) and (289) implies that $C^1_I = 0$ and $b_I = 0$. Further more from the definition of b_I comes that $b_I = ((\mathbf{H}^P)^{-1}\mathbf{H}^{G+YM}\vec{C}^2)_I$, so the solution is:

$$\tilde{C}_I(\underline{\Gamma}) = C^2_I, \tag{294}$$

where C_I^2 must satisfy the condition

$$H_{I'I}^{G+YM} C_I^2 = 0. \tag{295}$$

This is not a surprising result since if $L(v) = 0$ then substituting this into the constraints we obtain a theory completely equivalent to the electromagnetic field coupled to gravity. If we rewrite the scalar constraint of this theory in terms of the notation used in appendix A, we obtain the above condition.

What happens if $L(v) \neq 0$. In this case $C_I^2 = 0$ so that the solution does not become singular at $\Gamma = -a$. Substituting into (289) will yield the identity 0=0, so we must check (288). For $L(v) = \pm 1$ this will be singular so in this case $C_I^1 = 0$ and only the first term survives, but it will be zero too. Thus in this case the solution near $\Gamma = -a$ is zero in first order. For $|L(v)| > 1$ the condition (288) is also an identity. But in this case $\tilde{C}_I(-a) = 0$, so $b_I = 0$, thus

$$\tilde{C}_I(\underline{\Gamma}) = C_I^1 (\Gamma(v) + a)^{n_1}. \tag{296}$$

This solution tends rapidly to zero as $\Gamma \to -a$ (especially if $L(v)$ is large), so as we reach this limit, the amplitude of the solution coming from the $L(v) = 0$ case will become significantly larger. In fact the larger $L(v)$ is, the amplitude becomes much smaller in this region. So we can say that if $\Gamma(v) + a \approx 0$ (which - as we will see later - can be interpreted as the mass is about zero) states which for which $L(v) = 0$ have the highest probability while the larger $|L(v)|$, the smaller this probability will get.

These results show that in contrast to the Proca field, this theory provides us with the different amplitudes for different masses. However because the two theories are - in some aspect - very similar, it would be desirable to provide the solutions of this theory which can be identified as the solutions to the Proca field. The basic idea is very simple: we compare the two Hamiltonians. If we look at the matrix (286) in our differential equation, in the case $\Gamma = 0$ it will be the same as the Hamiltonian of the Proca field. So one just needs to imply the conditions

$$\frac{\delta^2}{\delta \Gamma(v)^2} \tilde{C}_{I'}(\underline{\Gamma}) \quad = \quad 0 \tag{297}$$

$$(\sum_{I'} H_{I'I}(\underline{\Gamma})\tilde{C}_{I'}(\underline{\Gamma}))_{|\Gamma=0} \quad = \quad 0 \tag{298}$$

The problem is that in this theory this will provide a distributional solution in the following sense. In the case of the Proca field the mass is fixed, which means that we are interested in solutions where Γ is constant. But now we have a differential equation so Γ is continuous. The way out of this is we say that in the interval $(-\epsilon, \epsilon)$ we solve (297), and outside this interval $\tilde{C}_{I'}(\underline{\Gamma})$ is zero. The required solution will the limit $\epsilon \to 0$. The reason for this strange behavior is that the equation we gained looks not like the Proca, but the linear combination of all the Procas.

6.6 Mass

In quantum field theory the mass is the coefficient of the term in the Hamiltonian which is quadratic in the boson field. However in this case we shall define the mass as an operator corresponding to the classical expression $\eta + a$. The reasons for us to do so are the following: First - as was shown at the end of section 6.2.2 - the expression $\eta + a$ corresponds exactly to the mass parameter of the Proca field (The term $(\eta + a)^2$ not only appears in front of the quadratic term of the bosonic field but also appears in the denominator of the kinetic term of the other scalar field). The second reason is that in this case we can simplify our analysis regarding the scalar-boson interaction. This new interpretation - as we will see - gives a better understanding of the mass generation in the Hamiltonian framework. Note also that the substitution $\eta = 0$ gives back the "original" mass.
Let $|\Psi\rangle := \sum_{\underline{\lambda}^\eta} \int d\underline{\Gamma}^\eta C_{\underline{\lambda}^\eta}(\underline{\Gamma}^\eta)|\underline{\Gamma}^\eta, \underline{\lambda}^\eta\rangle$ be a solution of the constraints. Then we can define the "mass operator" as

$$\hat{m}|\Psi\rangle = \frac{1}{\lambda} \arccos\Big(\frac{U_\eta(\lambda, v) + U_\eta^{-1}(\lambda, v)}{2}\Big)|\Psi\rangle = \sum_{\underline{\lambda}^\eta} \int d\underline{\Gamma}^\eta C_{\underline{\lambda}^\eta}(\underline{\Gamma}^\eta)\Gamma^\eta(v)|\underline{\Gamma}^\eta, \underline{\lambda}^\eta\rangle. \tag{299}$$

It is clear from the definition that this operator is self adjoint, thus it has real eigenvalues. Further more its spectrum is continuous. Its expectation value is

$$m(\Psi, v) \quad = \quad \sum_{\underline{\lambda}_1^\eta, \underline{\lambda}_2^\eta} \int d\underline{\Gamma}_1^\eta d\underline{\Gamma}_2^\eta C^*_{\underline{\lambda}^\eta}(\underline{\Gamma}_1^\eta) C_{\underline{\lambda}^\eta}(\underline{\Gamma}_2^\eta) e(\Gamma_2^\eta(v) + a)\langle\underline{\Gamma}_1^\eta, \underline{\lambda}_1^\eta|\underline{\Gamma}_2^\eta, \underline{\lambda}_2^\eta\rangle =$$

$$= \sum_{\underline{\lambda}^\eta} \int d\underline{\Gamma}^\eta |C_{\underline{\lambda}^\eta}(\underline{\Gamma}^\eta)|^2 e(\Gamma^\eta(v) + a), \tag{300}$$

thus for a graph γ we may define the mass as

$$m(\Psi, \gamma) = \sum_v m(\Psi, v) = \sum_{\underline{\lambda}^\eta} \int d\underline{\Gamma}^\eta |C_{\underline{\lambda}^\eta}(\underline{\Gamma}^\eta)|^2 \sum_v e^2(\Gamma^\eta(v) + a). \tag{301}$$

This means that $\Gamma(v) + a$ can be interpreted as "mass in a vertex". If we look at a state where all Γ are zero - the vacuum - we obtain states with mass ea. But one may ask whether this is an observable or not. If one checks the commutator of the constraints and $\hat{m}$ the only non-vanishing term will be the $[\hat{H}_P, \hat{m}]$ commutator, which is proportional to $X(v)$. This means that if take the subset of the solutions where $X(v)\Psi = 0$, the mass operator will be an observable. But if we look at the action of $X(v)$ in our new basis in (276) one will find that this is equivalent to the condition (297). So $\hat{m}$ is an observable if the solutions are those which are equivalent to the solutions of the Proca field. But one may say that there are other solutions as well, since one does not have to impose (298). The answer is that these states are special cases which are contained in the Proca solution. This is because in this case one has to solve $\sum_{I'} H_{I'I}(\underline{\Gamma})\tilde{C}_{I'}(\underline{\Gamma})$ for all Γ, which means that the solution will have to be in the kernel of all matrices appearing in $H_{I'I}$.

All in all the mass operator is an observable if if the solutions are those which are equivalent to the solutions of the Proca field. Since the Proca field did not have a potential term, the correspondence is correct only if we consider states where all Γ are zero (note that other states the mass operator is also an observable, but it describes interactions).

7 Conclusion and outlook

We have seen that it is possible to quantize the Proca-field in Loop Quantum Gravity, but one needs to introduce an extra scalar field so that the constraints of the theory become first class. Unfortunately this is necessary because the Dirac-brackets cannot be implemented on the quantum level. This problem also occurs when, after the theory has been quantized, we would want to recover the quantized version of the original Proca-field by gauge fixing, which leads us again to second class constraints. The quantized constraints can be solved (partially) if one generalizes the method introduce by Thiemann [33]. An interesting aspect is that the $m = 0$ case is not prohibited, it can be reached only asymptotically. In fact, the current form of theory does not say anything on the conditions regarding the mass, it has to be measured and put in by hand.

The case of spontaneous symmetry breaking results in a similar theory to the Proca field, even at the classical level. Here we have two scalar fields: one is completely analogous to the field introduce at the Proca field, while the other is the Higgs field. If the latter is constant, we obtain the constraints of the Proca field. To examine the properties of the mass operator, we introduce a new basis instead of the vertex functions (since these are eigenstates of the momentum operator). The by-product of this that it is possible to solve almost all of the constraints, and analyze some special solutions. We found that in contrast to the Proca-field, the state corresponding to $m = 0$ is part of the solution, and those have the largest amplitude that solve the constraints of the electromagnetic field coupled to gravity case. Further examination of the mass operator shows that it is self-adjoint, so its eigenvalues are real. There are special states for which this operator is an observable, that is it (weakly) commutes with all the constraints. This case has the interesting property that the Higgs field becomes a pure gauge, thus it cannot be considered as a physical field.

The main question is whether this method can be generalized to other cases, especially to the $SU(2)\times U(1)$ case. This looks problematic since then the action of the operators of the scalar field become more complicated, thus searching for eigenvalues of the mass seems hopeless. But examining the Salam-Weinberg model in Loop Quantum Gravity would provide a link with a theory that has been thoroughly tested. There is also a question of the positivity of mass. Though the corresponding operator is self-adjoint, there is no evidence that somehow negative eigenvalues are prohibited. If one looks at the results from a different perspective, one may ask if it was possible to generate mass by using other types of field instead of scalar fields? Can this field be gravitational field? In theory this can be done but there is no indication, either experimental or theoretical, that this is necessary. This is also true when one would consider a massive gravitational field.

8 Appendix A: The volume operator

Regularisation

As we have seen, the existence of the volume operator is crucial in order to define the scalar constraint. We will now derive it using the point-splitting formula of [50]. We will set $\beta = 1$, otherwise one has to multiply the final formula by $\beta^{3/2}$.
Let $R \subset \sigma$ be an open, connected region of σ. Since $E^a_J = \sqrt{q} e^a_J$ we have

$$\frac{1}{3!}\epsilon^{IJK}\epsilon_{abc}E^a_I E^b_J E^c_K = det(E^a_I) = sgn(det(E))det(q_{ab}), \tag{302}$$

thus the volume of a region R will be

$$V(R) = \int_R d^3x\sqrt{q} = \int_R d^3x\sqrt{\left|\frac{1}{3!}\epsilon^{IJK}\epsilon_{abc}E^a_I E^b_J E^c_K\right|}. \tag{303}$$

The next step is to smear and regulate this expression. For this let $\chi_\Delta(x,p)$ be the characteristic function in the coordinate x of a cube with center p, spanned by three vectors $\vec{\Delta}_i := \Delta_i \vec{n}_i(\Delta)$ where $\vec{n}_i$ is a normal vector and which has volume $vol = \frac{1}{\Delta_1\Delta_2\Delta_3}\det(\vec{n}_1, \vec{n}_2, \vec{n}_3)$ (we assume that the three normal vectors are right-oriented). In other words $\chi_\Delta(x,p) = \prod_{i=1}^3 \theta(\Delta_i/2 - |\vec{n}_i \cdot (\vec{x} - \vec{p})|)$ where $\theta(x)$ is the step function and $\cdot$ is the standard Euclidean scalar product.
Now consider the smeared quantity

$$\begin{aligned} E(p,\Delta,\Delta',\Delta'') := \frac{1}{vol(\Delta)vol(\Delta')vol(\Delta'')}\int_\sigma d^3x \int_\sigma d^3y \int_\sigma d^3z \chi_\Delta(p,x)\chi_{\Delta'}(2p,x+y) \times \\ \times\chi_{\Delta''}(3p,x+y+z)\frac{1}{3!}\epsilon^{IJK}\epsilon_{abc}E^a_I(x)E^b_J(y)E^c_K(z) \end{aligned} \tag{304}$$

Notice that if we take the limits $\Delta \to 0, \Delta' \to 0, \Delta'' \to 0$ in any combination we get back to (302) at the point p. This means that

$$V(R) = \lim_{\Delta\to 0}\lim_{\Delta'\to 0}\lim_{\Delta''\to 0}\int_R d^p\sqrt{|E(p,\Delta,\Delta',\Delta'')|}. \tag{305}$$

To define the corresponding operator we will use the same strategy which led to the electric flux operator. Let γ be a graph and in order to simplify the notation, let us subdivide each edge e with endpoints v, v' (which are vertices of γ) into two segments s, s' where $e = s \circ s'$ and have orientations such that s is outgoing at v and s' is outgoing at v'. This introduces new vertices $s \cap s'$ which we call "pseudo vertices" because they are not points of non-semianalyticity of the graph. Let $E(\gamma)$ be the set of all segments, but $V(\gamma)$ be the set of "real" vertices. Let us now evaluate the action of $\hat{E}^a_I(p,\Delta) := (1/vol(\Delta)) \int_\sigma d^3x \chi_\Delta(p,x) \hat{E}^a_I(x)$ on a function $f = p^*_\gamma f_\gamma$ cylindrical with respect to γ. We find that

$$\begin{aligned}\hat{E}^a_I(p,\Delta)f = \frac{il_P^2}{2vol(\Delta)} \sum_{e \in E(\gamma)} \int_0^1 dt \chi_\Delta(p, e(t)) \dot{e}^a(t) \times \\ \times \frac{1}{2} tr\left([h_e([0,t])\tau_I h_e([t,1])]^T \frac{\partial}{\partial h_e([0,1])}\right) f_\gamma\end{aligned} \tag{306}$$

where $e(t)$ is the parametrisation of the edge e. Here we used the following facts:

- $\{E^a_I, A^J_b\} = (\kappa/2)\delta(x-y)\delta^a_b\delta^J_I$
- The definition of the holonomy as the path-ordered exponential $\int_e A$ with the smallest parameter value to the left
- $\tau_I \in SU(2),\ \tau_J = -i\sigma_J$, where σ_I are the Pauli-matrices, so $[\tau_I/2, \tau_J/2] = \epsilon_{IJK}\tau_K/2$.
- We defined $tr(h^T \partial/\partial g) := h_{AB}\partial/\partial g_{AB}$ where $A, B, \ldots$ are SU(2) indices.

Of course the state on the right-hand side of (306) is well-defined in case of smooth connections. This is not problematic at the moment, since first we take the limits $\Delta \to 0$ etc. and then we see whether the result can be lifted to generalised connections. Now we have to calculate the action of $\hat{E}(p,\Delta,\Delta',\Delta'')$ on the state f. It is clear that we will have three kinds of terms. The first type comes from all three functional derivatives acting on f, the second comes when two functional derivatives act on f and one acts on the trace and finally when one functional derivative acts on f and the remaining two on the trace. Let us write the action of $\hat{E}(p,\Delta,\Delta',\Delta'')$ in the following form:

$$\begin{aligned}\hat{E}(p,\Delta,\Delta',\Delta'')f &= \left[\int_{[0,1]^3} dt dt' dt'' \epsilon_{abc} \dot{e}^a \dot{e}'^b \dot{e}''^c \times \right. \\ &\times \left. \chi(p,e)\chi(2p, e+e')\chi(3p, e+e'+e'')\hat{O}_{ee'e''}\right] f\end{aligned} \tag{307}$$

Here e, e', e'' are the parametrisation of the three edges (there are three integrals) and $\hat{O}_{ee'e''}$ contains the various functional derivatives (see details in [50]).
Now consider the function $x_{ee'e''}(t, t', t'') := e(t) + e'(t') + e''(t'')$. This has the interesting property that its Jacobian is given by

$$det\left(\frac{\partial(x^1_{ee'e''}, x^2_{ee'e''}, x^3_{ee'e''})(t, t', t'')}{\partial(t, t', t'')}\right) = \epsilon_{abc}\dot{e}^a\dot{e}'^b\dot{e}''^c$$

which is exactly the form of factor that enters the integral in (307) - this is why we have introduced the strange argument x+y+z. Now we use this for the following lemma:

Lemma 8.1 *For each triple of edges e, e', e'' there exists a choice of vectors $\vec{n}_i, \vec{n}'_i, \vec{n}''_i$ and a way to guide the limit $\Delta_i, \Delta'_i, \Delta''_i \to 0$ such that*

$$\int_{[0,1]^3} dtdt'dt''det\left(\frac{\partial(x^a_{ee'e''})(t, t', t'')}{\partial(t, t', t'')}\right)\chi(p, e)\chi(2p, e + e')\chi(3p, e + e' + e'')\hat{O}_{ee'e''} \qquad (308)$$

vanishes
a) if e, e', e'' do not all intersect p or
b) $det\left(\frac{\partial(x^a_{ee'e''})(t,t',t'')}{\partial(t,t',t'')}\right)_p = 0$ (which is a diffeomorphism invariant statement).
Otherwise it tends to $(1/8)sgn(det\left(\frac{\partial(x^a_{ee'e''})(t,t',t'')}{\partial(t,t',t'')}\right))_p\hat{O}_{ee'e''}(p)\prod_{i=1}^3 \Delta_i$

The proof is quite technical, one simply does a case subdivision and constructs the limit for each (see [50]). One may wonder whether is it a correct procedure that we adopt the regularisation to each triple of edges. The answer to this is yes since the classical expression does not depend on the way we regularise when we take the limit. The factor 1/8 appears in the final result because in the end one has to integrate a three dimensional Dirac delta which support has p as its endpoint.
Putting everything together our final result for the volume operator is

$$\hat{V}(R) = \frac{l_P^3}{8}\int_R d^3x \sum_{v\in V(\gamma)} \delta^3(p, v)\hat{V}_{\gamma,v}$$

$$\hat{V}_{\gamma,v} = \sqrt{\left|\frac{i}{8\cdot 3!}\sum_{e,e',e''\in E(\gamma), e\cap e'\cap e''=v} s(e, e', e'')\epsilon_{IJK}X^I_e X^J_{e'} X^K_{e''}\right|} \qquad (309)$$

where $s(e,e',e'')$ is the signum of the determinant and
$X_e^I = X^I(h_e([0,1])) = tr((\tau_I h_e([0,1]))^T \frac{\partial}{\partial h_e([0,1])})$ etc. are the right invariant vector fields in the τ_I direction of $SU(2)$.

Properties of the volume operator

Cylindrical consistency: Because the regularisation was adapted to the state it acts on we obtained a family of operators $\hat{V}_\gamma$. Cylindrical consistency means that if $\gamma \subset \gamma'$ then $(\hat{V}_{\gamma'})_\gamma = \hat{V}_\gamma$. It is easy to prove this if one considers the fact that γ can be obtained from γ' by a finite series of steps consisting of the the following basic ones:
1) Remove an edge of γ'.
2) Join two edges e', e'' such that $e' \cap e''$ is a point of analyticity to a new edge $e = e' \circ (e'')^{-1}$.
3) Reverse the orientation of the edge.

In case 1) it is trivial that $X_e^I f_\gamma = 0$ for any function that is cylindrical with respect to γ, so the terms involving e drop out. In case 2) v will be a pseudo vertex of γ, thus $\hat{V}_\gamma$ does not have a term corresponding to v. On the other hand - because v is a pseudo vertex - it is a bivalent vertex in γ' so the corresponding term in $\hat{V}_{\gamma'}$ drops out. Likewise if v is a vertex for γ at which the outgoing edge is incident then from right invariance we obtain $X_e = X_{e'\circ(e'')^{-1}} = X_{e'}$ so at vertices that belong to both γ and γ' the corresponding vertex operators coincide. Finally case 3) is excluded because for our choice of orientation. To conclude there exists an well defined operator on $\mathcal{H}$.
Diffeomorphism covariance: Though during the process of regularisation we have broken diffeomorphism covariance, the final result is manifestly covariant.
Symmetry, positivity and self-adjointness: Notice that iX_e is symmetric on $\mathcal{H}_\gamma$ with respect to the measure μ_γ, because the Haar measure is right invariant. This means that the projections $\hat{V}_\gamma$ are symmetric and since the volume operator leaves the underlying graph invariant this is enough to show that $\hat{V}$ is also symmetric. Further more $\hat{V}_\gamma$ are positive semidefinite so that $\hat{V}$ is a densely defined, positive semidefinite and symmetric operator, which means it has self adjoint extensions.
Discreetness and anomaly freeness: The volume operator has the important property that for an arbitrary spin network function it does not change neither the graph, nor the labels,

only the intertwiner. This means that the matrix

$$(V_{\gamma,\vec{\pi}})_{\vec{I},\vec{I'}} =< T_{\gamma,\vec{\pi},\vec{I}}|\hat{V}|T_{\gamma,\vec{\pi},\vec{I'}} >$$

is finite-dimensional (there are only finite intertwiners that are compatible with γ and $\vec{\pi}$), positive and symmetric. Moreover since $\hat{V}_{\gamma,v}$ involves those edges which are incident at v, we find $[\hat{V}_{\gamma,v}, \hat{V}_{\gamma,v'}] = 0$, thus each $\hat{V}_{\gamma,v}$ can be diagonalised separately. This property is important for two reasons: first one can show that the operator is anomaly free, and second that the entire spectrum of the operator is discrete.

Matrix elements: Unfortunately the volume operator cannot be diagonalised in closed form. The reason for this that the operator $\hat{Q}_{v,\gamma}$ - related to the volume operator through $\hat{V}_{v,\gamma} = \sum_v \sqrt{|\hat{Q}_{v,\gamma}|}$ - is a third order polynomial of the right invariant vector fields. While one can calculate the matrix elements of $\hat{Q}_{v,\gamma}$ in the spin network basis using the quantum mechanics of n_v angular momentum operators (n_v is the valance of the vertex v), the resulting matrix has no special symmetries, thus its eigenvalues cannot be calculated for general $\vec{\pi}$. This poses a big problem since the volume operator is a key element of the Hamiltonian constraint. There are two ways present at the literature of how one copes with this problem: exact solutions for low n_v and coherent states.

9 Appendix B: Simplification of the notation

Here we introduce a notation which simplifies the scalar constraint. Though this is not necessary, it is useful to make the constraint more transparent. Let us introduce a multi-index I for the indices $s, f, \bar{\Gamma}, \underline{\bar{\delta}}$ so that a type of expression $\langle s'|\langle f'|\langle\bar{\Gamma}, \underline{\bar{\delta}}|\hat{X}|\bar{\Gamma}, \underline{\bar{\delta}}\rangle|f\rangle|s\rangle$ will be denoted as $X_{I'I}$. Now consider those terms that do not contain U_η or $\hat{X}_\eta$. These are $\hat{H}_{grav}$ and $\hat{H}_{YM}$, the Hamilton operator of the gravitational and Maxwell field. So in our new notation the contribution of these terms to the constraint equations will be the following:

$$\begin{aligned}\sum_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}\int d\Gamma(v)'\int d\bar{\Gamma}(v)'(H^G_{ss'}\delta_{ff'}\delta_{\underline{\lambda\lambda}'}\delta_{\underline{\bar{\delta}\bar{\delta}}'}\delta(\Gamma(v)',\Gamma(v))\delta(\bar{\Gamma}(v)',\bar{\Gamma}(v))+\\ +H^{YM}_{ff'}G^1_{ss'}\delta_{\underline{\lambda\lambda}'}\delta_{\underline{\bar{\delta}\bar{\delta}}'}\delta(\Gamma(v)',\Gamma(v))\delta(\bar{\Gamma}(v)',\bar{\Gamma}(v)))C_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}(\underline{\Gamma}',\underline{\bar{\Gamma}}')=\\ =\sum_{I'}H^{G+YM}_{I'I}(v)C_{I'}(\underline{\Gamma}),\end{aligned} \tag{310}$$

where we performed the integration and sum on the variables related to the scalar fields. The terms containing U_η or $\hat{X}_\eta$ will be treated as follows: we shall write the dependence of these fields explicitly, while other expressions will be denoted (using the short notation) as $O^1_{I'I}$ etc. For example the contribution of the potential term $\hat{H}_{pot}$ will be denoted as follows:

$$\begin{aligned}\sum_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}\int d\Gamma(v)'\int d\bar{\Gamma}(v)'\frac{1}{4}N(v)\mu\langle s|\hat{V}|s'\rangle\delta_{ff'}\delta(\bar{\Gamma},\bar{\Gamma}')\delta_{\underline{\bar{\delta}\bar{\delta}}'}\delta_{\underline{\lambda\lambda}'}\frac{1}{\lambda^2}\times\\ \times C_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}(\underline{\Gamma}',\underline{\bar{\Gamma}}')\langle\Gamma,\underline{\lambda}|\arccos\Big(\frac{U_\eta(\lambda,v)+U^{-1}_\eta(\lambda,v)}{2}\Big)^2\times\\ \times\Big[\frac{1}{\lambda}\arccos\Big(\frac{U_\eta(\lambda,v)+U^{-1}_\eta(\lambda,v)}{2}\Big)+2a\Big]^2|\Gamma',\underline{\lambda}'\rangle=\\ =\sum_{I'}H^{pot}_{I'I}(v)C_{I'}(\underline{\Gamma})\Gamma(v)^2(\Gamma(v)+2a)^2\end{aligned} \tag{311}$$

The derivative term $\hat{H}_{der}(\Theta)$ contains the operator U_η, so we have

$$\sum_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}\int d\Gamma(v)'\int d\bar{\Gamma}(v)'\frac{1}{2}\frac{N(v)}{E(v)^2}\times$$

$$\times\langle\Gamma,\underline{\lambda}|\Big[\frac{1}{\lambda}\arccos\Big(\frac{U_\eta(\lambda,v)+U_\eta^{-1}(\lambda,v)}{2}\Big)+a\Big]^2|\Gamma',\underline{\lambda}'\rangle\times$$
$$\times\sum_{v(\Delta)=v(\Delta')=v}\frac{1}{\delta^2}\langle\bar{\Gamma},\underline{\bar{\delta}}|[U_\Theta(\delta,s_n(\Delta))\underline{h}_{s_n}U_\Theta(\delta,v)^{-1}-1]\times$$
$$\times[U_\Theta(\delta,s_r(\Delta'))\underline{h}_{s_r}U_\Theta(\delta,v)^{-1}-1]|\bar{\Gamma}',\underline{\bar{\delta}}'\rangle\langle s|\hat{G}_2^{nr}(v)|s'\rangle C_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}(\underline{\Gamma}',\underline{\bar{\Gamma}}')=$$
$$=\sum_{I'}H_{I'I}^{der1}C_{I'}(\underline{\Gamma})(\Gamma(v)+a)^2 \qquad (312)$$

The other derivative term is a bit trickier since the U_η is evaluated in different vertices. We have

$$\sum_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}\int d\Gamma(v)'\int d\bar{\Gamma}(v)'\frac{1}{2}\frac{N(v)}{E(v)^2}\sum_{v(\Delta)=v(\Delta')=v}\frac{1}{\lambda^2}\langle\Gamma,\underline{\lambda}|[U_\eta(\lambda,s_n(\Delta))U_\eta^{-1}(\lambda,v)-1]\times$$
$$\times[U_\eta(\lambda,s_r(\Delta'))U_\eta^{-1}(\lambda,v)-1]|\Gamma',\underline{\lambda}'\rangle\langle s|\hat{G}_2^{nr}(v)|s'\rangle C_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}(\underline{\Gamma}',\underline{\bar{\Gamma}}')=$$
$$=\sum_{I'}(H_{I'I}^A\exp(-2i\lambda\Gamma(v))-H_{I'I}^B\exp(-i\lambda\Gamma(v))+H_{I'I}^C)C_{I'}(\underline{\Gamma}), \quad (313)$$

where the values $\exp(i\lambda\Gamma(s_r(\Delta')))$ etc. have been assimilated in the coefficients $H_{I'I}^A$ etc. since our differential equation will depend only variables in vertex v.
The last contribution is the momentum term. It is convenient to use equation (282) so that one can simplify this expression in the following way:

$$\sum_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}\int d\Gamma(v)'\int d\bar{\Gamma}(v)'\frac{1}{2}\frac{N(v)}{E(v)^2}((\langle\Gamma,\underline{\lambda}|X_\eta(v)^2|\Gamma',\underline{\lambda}'\rangle\delta_{\underline{\delta}\underline{\delta}'}\delta(\bar{\Gamma}(v)',\bar{\Gamma}(v))+$$
$$+\langle\Gamma,\underline{\lambda}|\Big[\frac{1}{\lambda}\arccos\Big(\frac{U_\eta(\lambda,v)+U_\eta^{-1}(\lambda,v)}{2}\Big)+a\Big]^{-2})|\Gamma',\underline{\lambda}'\rangle\times$$
$$\times\langle\bar{\Gamma},\underline{\bar{\delta}}|X_\Theta(v)^2)|\bar{\Gamma}',\underline{\bar{\delta}}'\rangle\langle s|\hat{G}_1(v)|s'\rangle\delta_{ff'}C_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}(\underline{\Gamma}',\underline{\bar{\Gamma}}')=$$
$$\sum_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}\int d\Gamma(v)'\int d\bar{\Gamma}(v)'\frac{1}{2}\frac{N(v)}{E(v)^2}((\langle\Gamma,\underline{\lambda}|X_\eta(v)^2|\Gamma',\underline{\lambda}'\rangle+$$
$$+\frac{L(v)^2}{(\Gamma(v)+a)^2}\delta(\Gamma(v)',\Gamma(v))\delta_{\underline{\lambda}\underline{\lambda}'})\langle s|\hat{G}_1(v)|s'\rangle\delta_{\underline{\delta}\underline{\delta}'}\delta(\bar{\Gamma}(v)',\bar{\Gamma}(v))\delta_{ff'}C_{s',f',\underline{\lambda}',\underline{\bar{\delta}}'}(\underline{\Gamma}',\underline{\bar{\Gamma}}')=$$
$$=\sum_{I'}((\lambda_v-i\frac{\delta}{\delta\Gamma(v)})^2+\frac{L(v)^2}{(\Gamma(v)+a)^2})H_{I'I}^P C_{I'}(\underline{\Gamma}), \quad (314)$$

where $L(v)=\sum_v l_e$ is the sum of integers coming from the flux network state.

10 Appendix C: Solving a system of second order differential equation

Here we describe the method to solve a system of differential equation of the form

$$\ddot{\vec{c}}(t) = \mathbf{H}(t)\vec{c}(t), \tag{315}$$

where $\mathbf{H}(t)$ is a $N \times N$ matrix.

The method is similar to the one used in cases of path ordered integration. First we integrate the equation:

$$\dot{\vec{c}}(t) = \int_0^t dt_1 \mathbf{H}(t_1)\vec{c}(t_1) + \vec{c}_1, \tag{316}$$

where

$$\vec{c}_1 = \dot{\vec{c}}(0).$$

Another integration yields

$$\vec{c}(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \mathbf{H}(t_2)\vec{c}(t_2) + \vec{c}_1 t + \vec{c}_0, \tag{317}$$

where

$$\vec{c}_0 = \vec{c}(0).$$

Now we iterate this equation and arrive to the result

$$\begin{aligned} \vec{c}(t) = &\Big(1 + \sum_{j=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{2j-1}} dt_{2j} \mathbf{H}(t_2)\mathbf{H}(t_4)\ldots\mathbf{H}(t_{2j})\Big)\vec{c}_0 + \\ &+\Big(t + \sum_{j=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{2j-1}} dt_{2j} \mathbf{H}(t_2)\mathbf{H}(t_4)\ldots\mathbf{H}(t_{2j})t_{2j}\Big)\vec{c}_1 \end{aligned} \tag{318}$$

Though our case ($\mathbf{H}(t) \approx \mathbf{C}\frac{1}{(t+2)^2}$ where $\mathbf{C}$=const.) looks simple, the above integral cannot be calculated in closed form since even for j=2 we must calculate the integral of $ln(t)/(''Polynomial \quad of \quad t'$

References

[1] G. Helesfai, *The Proca-field in Loop Quantum Gravity* Classical and Quantum Gravity 24:2861-2878,2007 [gr-qc/0605048]

[2] G. Helesfai, *Spontaneous symmetry breaking in Loop Quantum Gravity* Classical and Quantum Gravity 25:235010,2008 [arXiv:0806.3356]

[3] Robert M. Wald *General Relativity*, The University of Chicago Press, Chicago 60637

[4] M. B. Green, J. Schwarz, E. Witten *Superstring Theory*, Cambridge University Press, Cambridge, (1987)

[5] S. A. Huggett and K. P. Tod, *An introduction to twistor theory*, London Mathematical Student Texts 4 (New edition, 1994).

[6] D.Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, *Progress Toward A Theory Of Supergravity*, *Phys. Rev.* D **13** (1976) pp 3214-3218.

[7] G. Brightwell, R. Gregory, *Structure of random discrete spacetime*, *Phys. Rev. Lett.* 66, 260 - 263, (1991)

[8] A. Connes *Noncommutative Geometry*, Academic Press (1994)

[9] R. Arnowitt, S. Deser, C. W. Misner *Gravitation: An Introduction to current Research*, L. Witten (ed.), Wiley, New York, (1962)

[10] P. A. M. Dirac *Lectures on Quantum Mechanics* Belfer Graduate School of Science, Yeshiva University Press, New York, (1964)

[11] A. Ashtekar *New Hamiltonian formulation of general relativity*, Phys. Rev. D36 (1987) 1587

[12] T. Thiemann *Modern Canonical Quantum General Relativity*, Cambridge University Press

[13] M. Bojowald *Loop Quantum Cosmology I: Kinematics*, *Class.Quant.Grav.* **17** (2000) 1489-1508

[14] C. Rovelli and T. Thiemann *The Immirzi parameter in quantum general relativity Phys. Rev.* **D57** (1998) 1009-14. [gr-qc/9705059]

[15] A. Ashtekar and C.J. Isham, *Representations of the holonomy algebras of gravity and non-abelian gauge theories*, *Class. Quantum Grav.* **9**, 1433-85, (1992)

[16] J. Barbero *Solving the Constraints of General Relativity*, gr-qc/9411016

[17] T. Thiemann *A Length Operator for Canonical Quantum Gravity*, Journ. Math. Phys. 39 (1998) 3372-2292, [gr-qc/9606092]

[18] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, T. Thiemann, *Quantization for diffeomorphism invariant theory of connections with local degrees of freedom*, Journ. Math. Phys. 36 (1995) 6456-6493,[gr-qc/9504018]

[19] A. Ashtekar, J. Lewandowski *Quantum Theory of Geometry II: Volume operators*, Adv. Theo. Math. Phys. 1 (1997) 388-429

[20] Carlo Rovelli, L. Smolin *Discreteness of Area and Volume in Quantum Gravity*,

[21] C. Rovelli and L. Smolin *Loop representation of quantum general relativity*, *Nucl. Phys. B* **331** (1), 80-152, 1990. *Nucl. Phys.* **B442** (1995) 593-622. Erratum: **B456** (1995) 753. [gr-qc/9411005]

[22] P. Hajicek and K. Kuchar *Constraint quantisation of parametrised relativistic gauge systems in curved space-times*, *Phys. Rev.* **D41** (1990) 1090-104.

[23] P. Hajicek and K. Kuchar *Transversal affine connection and quantisation of constrained systems*, *J. Math. Phys.* **31** (1990) 1723-32.

[24] C. Rovelli and L. Smolin *Spin Networks and Quantum Gravity*, *Phys. Rev.* D **52**, 5743-5759, (1995)

[25] J. C. Baez *Spin Networks in Nonperturbative Quantum Gravity*, [gr-qc/9504036]

[26] Weinberg S 1995 *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, United Kingdom)

[27] A. Ashtekar 1987 *New Hamiltonian formulation of general relativity Phys. Rev.* D **36** 1587

[28] Henneaux M and Teitelboim (1992) *Quantisation of Gauge Systems* (Princeton University Press, Princeton, New Jersey)

[29] Hong S, Kim Y, Park Y and Rothe K D (2002) *Symplectic embedding and Hamilton-Jacobi analysis of Proca model Mod. Phys. Lett.* **A17** 435-451.

[30] R. Banerjee and J. Barcelos-Neto (1997) *Hamiltonian embedding of the massive Yang-Mills theory and the generalized Stckelberg formalism Nucl.Phys. B* **499** 453

[31] A. Ashtekar, J. D. Romano and R. S. Tate *New variables for gravity: Inclusion of matter* (1989) *Phys. Rev.* D **40** 2572

[32] T. Thiemann (1998) *Quantum Spin Dynamics (QSD) Class. Quant. Grav.* **15** 839, [gr-qc/9606089]

[33] T. Thiemann (1998) *Quantum Spin Dynamics (QSD) II Class. Quant. Grav.* **15** 875, [gr-qc/9606090]

[34] T. Thiemann (1998) *QSD V : Quantum Gravity as the Natural Regulator of Matter Quantum Field Theories Class. Quant. Grav.* **15** 1281, [gr-qc/9705019]

[35] T. Thiemann (1998) *Quantum Spin Dynamics (QSD): VI. Quantum Poncare Algebra and a Quantum Positivity of Energy Theorem for Canonical Class. Quant. Grav.* **15** 1463, [gr-qc/9705020]

[36] T. Thiemann 2000 *Introduction to Modern Canonical Quantum General Relativity Preprint* [gr-qc/0110034]

[37] A. Ashtekar and J. Lewandowski 2004 *Background independent quantum gravity: A status report Class. Quant. Grav.* **21** R53

[38] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao and T. Thiemann 1995 *Quantization of Diffeomorphism Theories of Connection with Local Degrees of Freedom J. Math. Phys.* **36** 6456

[39] R. Borissov, *Regularization of the Hamiltonian constraint and the closure of the constraint algebra*, *Phys. Rev.*, D **55**, 2059-2068, (1997)

[40] A. Ashtekar, C. Rovelli and L. Smolin, *Weaving a classical geometry with quantum threads*, *Phys. Rev. Lett.*, **69**, 237-240, (1992).

[41] J. Alfaro, H. A. Morales-Tecotl and L. F. Urrutia *Loop quantum gravity and light propagation* 2002 *Phys. Rev.* D **65** 103509

[42] M. Varadarajan 2000 *Fock representations from U(1) holonomy algebras Phys. Rev.* D **61** 104001

[43] M. Varadarajan 2001 *Photons from quantized electric flux representations Phys. Rev.* D **64** 104003

[44] H. A. Morales-Tcotl and C. Rovelli *Fermions in quantum gravity*, *Phys. Rev. Lett.*, **72**, 3642-45, (1994)

[45] T. Thiemann 1997 *Kinematical Hilbert Spaces for Fermionic and Higgs Quantum field Theories Preprint* HUTMP-97/B-364

[46] A. Ashtekar, J. Lewandowski and H. Sahlmann 2003 *Polymer and Fock representations for a Scalar field Class. Quant. Grav.* **20** L11-1

[47] W. Kaminski, J. Lewandowski and M. Bobienski 2005 *Background independent quantizations: the scalar field I Preprint* [gr-qc/0508091]

[48] J. M. Pons 1996 *Plugging the Gauge-Fixing Into the Lagrangian Int. J. Mod. Phys.* A11 **975**

[49] R. Gambini and J. Pullin 2005 *Consistent Discretization and Loop Quantum Geometry Phys. Rev. Lett.* **94** 101302

[50] T. Thiemann, *Closed formula for the matrix elements of the volume operator in canonical quantum gravity J. Math. Phys.* **39** (1998) 3347-71. [gr-qc/9606091]

Summary

In this thesis we described the method of quantizations of massive vector fields in the framework of loop quantum gravity. The first four sections give a summary about the tools of loop quantum gravity, which is a non-perturbative, diffeomorphism invariant, canonical quantization of general relativity. In the last two sections we apply these methods to the Proca field and spontaneous symmetry breaking. It turns out, that in the case of the Proca-field we experience difficulties at the 3+1 decomposition: the theory has a second class constraint algebra. While the methods of loop quantum gravity can be applied in this case too (one has to use Dirac brackets instead of the Poisson bracket), it is much easier to use a method called symplectical embedding, where first class constraint algebra is obtained by the introduction of an extra scalar field. Mass will be a parameter of the theory , but the case $m = 0$ can be reached only asymptotically. If we want to eliminate the scalar field from the theory after quantization by gauge fixing, we will again face the problems of a second class constraint algebra. Problems regarding the Dirac bracket also arises during the definition of elementary operators, because we have to use it or else the quantization would be inconsistent.

In the last section mass was generated by spontaneous symmetry breaking and the results were compared to the ones of the Proca field. The two theories have similar properties even at the classical level. The main difference is an extra scalar field and a potential term, and the mass will be a field. Quantization is almost the same for both theories except now the mass will be an operator. With the help of a new basis - which was defined to be eigenvalues of the multiplication operator - the constraints have been cast into a finite system of differential equations and we could analyze the scalar field dependence of the theory. We were able to (partially) solve the constraints and examine the states corresponding to the zero mass. We found that there exist non-singular, non-trivial solutions in this region where the only non-zero case will be the one that solves the constraints of the electromagnetic field coupled to gravity case. The spectrum of the mass operator is continuous (though it has a discrete structure due to the discreteness of the coefficients) and real, though not necessarily positive. The most interesting result is that there exist a subspace of the solution where the mass operator is an observable (it weakly commutes with the constraints), but in this case both scalar fields are pure gauge.

Összefoglalás

A doktori disszertáció a tömeges vektormezők kvantálását vizsgálja a kanonikus kvantumgravitáció keretein belül. Az első négy fejezetben a kanonikus kvantumgravitáció elméletébe nyújtunk betekintést, ami egy kanonikus, nem perturbatív, kovariáns kvantálása az általános relativitás elméletnek. Az utolsó két fejezetben tárgyaljuk a Proca-mező valamint a spontán szimmetria sértés elméletét ezen módszerek segítségével. Kiderül, hogy a Proca-mező esetében már a 3+1-es felbontásnál nehézségekbe ütközünk: az elmélet kényszerei másodfajúak. Bár a kanonikus kvantumgravitáció módszerei ebben az esetben is alkalmazhatóak - a Poisson-zárójeleket Dirac-zárójelekre kell cserélni -, sokkal célszerűbb azonban az ún. szimplektikus beágyazást alkalmazni, aminek során egy extra skalártér bevezetésével elsőfajú kényszereink lesznek. A tömeg az elmélet egy paramétere lesz, viszont az $m = 0$ esetet csak aszimptotikusan kapjuk vissza. Ha kvantálás után eliminálni akarjuk a skalárteret, akkor ugyanúgy a másodfajú kényszerek problémájával szembesülünk, mint a 3+1-es felbontásnál (a mértékrögzítés miatt). A Dirac-zárójel problémája az operátorok definiálásánál is felmerül, hiszen e nélkül inkonzisztens lenne ezek definíciója.
Az utolsó fejezetben a tömeget spontán szimmetria sértés segítségével generáltuk, és a kapott eredményeket összehasonlítottuk a Proca-mező eredményeivel. Már a klasszikus elméletben sok hasonlóság van a két megközelítésben; a fő különbség egy extra skalártér és a potenciál, valamint a Proca-mező tömege helyén itt egy mező szerepel. A kvantálás hasonlóképpen történt mindkét esetben, csak itt a tömeg operátor lett. Egy új bázis segítségével - amik a szorzó operátor sajátértékei - a kényszereket sikerült egy véges differenciálegyenletrendszer formájaban felírni, melynek során az elmélet skalártér függését vizsgáltuk. Ebben a bázisban sikerült részben megoldani a kényszereket és megvizsgálni a nulla tömeghez tartozó állapotokat. Azt találtuk, hogy léteznek nemtriviális és nem szinguláris megoldások ebben a tartományban, melyek közul $m = 0$ esetén csak az lesz nem zérus, ami megoldása az "elektromágneses tér csatolva a gravitációhoz" elmélet kényszerének. A tömegoperátor sajátértékei folytonosak (bár van egy diszkrét struktúrájuk az együttható mátrixok diszkrét voltából fakadóan) és valósak, viszont nem feltétlenül pozitívak (ehhez extra input kell). Az egyik legérdekesebb eredmény, hogy létezik egy részhalmaza a megoldásoknak, ahol a tömegoperátor megfigyelhető mennyiség (a kényszerekkel gyengén kommutál), de ebben az esetben mindkét skalártér puszta gauge.